Computer Science 2405
March 23, 2020

Happy Monday, March 23.

We are about to begin to look at ways of counting, especially where there are a large number of things to count or where we need a general formula. The mathematical area that is concerned with counting is called *combinatorics*.

A common theme in combinatorics is the very large numbers that occur. Because the numbers are so large, it is typically infeasible to run a computer program that searches through a whole collection.

It is *very important* that you work the exercises as you go. If you do not understand something, ask about it.

To get started, read Rosen, section 6.1.1 (Counting: Introduction)

Basic counting principles

The lecture covers basic counting rules are the fundamental tools that we will use for counting, with examples of each. Be careful to notice the restrictions on their use.

Read Rosen, section 6.1.2.

> **Product rule**
> Suppose that you will make two choices, one after another, where there are m ways to make the first choice and n ways to make the second choice. Then there are mn ways to make the combination of the two choices.

The product rule is versatile. A key restriction is that the outcome of the first choice must not influence the number of options that are available for the second choice. The two choices are made independently of one another.

Example. In an early version of the programming language Basic, a variable name needs to be two characters long, where the first character is a letter and the second is a digit. For example, $x1$ and $b5$ are variable names. How many allowed variable names are there?

Answer. There are 26 ways to select a letter and 10 ways to select a digit. So there are $(26)(10) = 260$ ways to choose a variable name. Equivalently, there are 260 variable names.

Example. A spreadsheet has 20 rows and 15 items per row. How many total items are there?

Answer. Counting items is the same as determining how may ways there are to select an item. To select an item, first select a row then select an item in that row. There are 20 ways to select a row and 15 ways to select an item in the row, so there are (20)(15) = 300 total items.

Work homework exercise 1 in exercise set 4.

<u>**Generalized product rule**</u> Suppose that you will make k choices, one after another, where there are n_1 ways to make the first choice, n_2 ways to make the second choice, ... , n_k ways to make the kth choice. Then there are $n_1 n_2 \cdots n_k$ ways to make the combination of the k choices.

Example. A bit string is a string of 0's and 1's. How many bit strings are there of length 3?

Answer. There are 3 choices: the first bit, the second bit and the third bit. There are 2 ways to make each choice. So there are (2)(2)(2) = 8 ways to choose a bit string of length 3.

Example. A license plate in a particular state has 3 letters followed by 3 digits. How many different license plates are there? Assume that there are 26 letters.

Answer. There are (26)(26)(26) = 17576 ways to choose the three letters and (10)(10)(10) = 1000 ways to select the three digits. So there are 17576 + 1000 = 18576 total ways to choose a license plate.

Example. Let $A = \{1, 2, 3, 4\}$ and $B = \{a, b, c\}$. How many different functions are there with domain A and codomain B? (If $f : A \to B$, then A is the domain of f and B is the codomain of f.)

Answer. We need to choose a value for each of $f(1)$, $f(2)$, $f(3)$ and $f(4)$. That is a sequence of 4 choices. Each choice can be made in 3 ways (a, b or c). So there are $(3)(3)(3)(3) = 3^4 = 81$ ways to choose a function with domain A and codomain B.

Example. Suppose A and B are finite sets where $|A| = m$ and $|B| = n$. How many different functions are there with domain A and codomain B?

Answer. We need to choose a value for $f(x)$ for each of m values of x. So there is a sequence of m choices to make. Each choice can be made in n ways. The total number of ways to choose a function $f : A \to B$ is $(n)(n)\cdots(n) = n^m$.

The previous two examples illustrate a general principle to help you derive a general formula. Look at examples of fixed sizes. Then generalize.

<u>Repetition rule</u> This is a case of the product rule. If you buy m things and each thing costs n, the total cost is mn.

Example. What is the value of k when the following program is finished?

```
k = 0;
for i = 1 to m
   for j = 1 to n
      k = k + 1;
```

Answer. Each time the inner loop is done, it adds n to k. Since the inner loop is done m times, a total of mn is added to k. So $k = mn$ at the end.

Sometimes, the key to a counting problem is to look at the problem the right way.

Example. Suppose that A is a finite set where $|A| = n$. How many subsets does A have?

Answer. Let's start by looking at a small example, $A = \{1, 2, 3\}$.

There is a useful for representing a set of small integers like A in a computer program. The idea is represent set $S \subseteq A$ as a sequence of bits, where the first bit tells you whether $1 \in S$, the second bit tells you whether $2 \in S$ and the third bit tells you whether $3 \in S$. Taking 0 to mean no and 1 to mean yes, bit-string 011 indicates set $\{2, 3\}$ and 000 indicates the empty set. (This representation is very compact and efficient to use for sets of small integers. You can represent a subset of $\{0, 1, \ldots, 63\}$ using only one 64-bit word.)

Thinking in terms of bit-strings, we see that the number of subsets of A is the same as the number of bit-strings of length n. You choose a length n bit-string by making n binary decisions (0 or 1). The product rule tells you that there are 2^n bit strings of length n. So a set of size n has 2^n different subsets.

Work homework exercises 2, 4, 5, 9a, 11, 12 in exercise set 4. For problem 5, first write down all of the bit strings of length 3 that start and end with 1. Then do the same for length 4. There is only one way to choose the first and last bit. How many binary choices are left?

For problem 12, let e be the number of employees. Write an equation that summarizes the facts that are given. Solve the equation for e.

<u>Sum rule</u>
Suppose that you have two disjoint finite sets A and B, where $|A| = m$ and $|B| = n$. You make a choice: either select a member of A or select a member of B (but not both). There are $m + n$ ways to choose.

Example. How many ways are there to select either a letter or a digit?

Answer. 26 + 10 = 36.

Example. You must choose exactly one project to do. There are two lists of projects to select from. The first list has 5 projects and the second list has 9 projects. No project occurs on both lists. How many choices to you have?

Answer. There are 5 + 9 = 14 choices.

Example. Looking again at an early version of Basic, variable names are allowed to have two forms: either just a letter of a letter followed by a digit. How many different variable names are there?

Answer. To write a variable name, you have a choice of a one-letter variable or a name consisting of a letter and a digit. There are 26 ways to choose a one-letter name and 260 ways to choose a two-symbol name. So there are 26 + 260 = 286 variable names.

Be careful with the sum rule. Suppose that $A = \{1, 2, 3, 4\}$ and $B = \{3, 4, 5, 6\}$. If you can choose a member of A or a member of B, there are 6 ways to choose, not 8. In general, the number of choices is $|A \cup B|$.

Sequencing rule
This is a case of the sum rule. If you buy two things, one costing m and the other costing n, the total cost is $m + n$.

Example. What is the value of k when the following program is finished?

```
k = 0;
for i = 1 to m
   k = k + 1;
for j = 1 to n
   k = k + 1;
```

Answer. The first loop adds m to k. The second loop adds n to k. So k is incremented $m + n$ times, and $k = m + n$ at the end.

Overcounting

Read Rosen, section 6.1.4.

Sometimes the easiest way to count something is to overcount and then compensate for the overcounting.

Subtraction rule
This is a generalization of the sum rule that is also called the *inclusion-exclusion* rule.
Suppose that you have two finite sets A and B, where $|A| = m$ and $|B| = n$. Sets A and B are not necessarily disjoint. Say that $|A \cap B| = e$.
You make a choice: either select a member of A or select a member of B (but not both). There are $m + n - e$ ways to choose.

Example. How many bit strings of length 8 either start with 00 or end with 11?

Answer. To select such a bit string, your first decision is whether to make the bit string start with 00 or to make it end with 11. In each case, there are only 6 remaining bits to choose. So there are $2^6 = 64$ bit strings of length 8 that begin with 00 and $2^6 = 64$ bit strings of length 8 that end with 11. The sum rule would suggest that there are $64 + 64 = 128$ choices.

But that is not correct because some bit strings of length 8 both begin with 00 and end with 11, and those have been counted twice. They have the form 00xxxx11 where each x is either 0 or 1. There are $2^4 = 16$ of those.

So there are $64 + 64 - 16 = 112$ bit strings of length 8 that either begin with 00 or end with 11.

Example. You have 300 plastic jars of m&ms. 150 of the jars contain at least one red m&m, 105 contain at least one blue m&m and 80 contain both red and blue m&ms. How many of the jars contain neither red nor blue m&ms?

Answer. Let J be the set of all 300 jars, R be the set of jars that contain at least one red m&m and B be the set of jars that contain at least one blue m&m. Finally, let N be the set of jars that contain neither red nor blue m&ms.

At this point, draw a Venn diagram showing a large circle for set J and two overlapping circles inside the large one; label those overlapping circles R and B. The intersection of R and B is the set of jars that contain both red and blue m&ms. We know that

$$\begin{aligned} |J| &= 300, \\ |R| &= 150, \\ |B| &= 105, \\ |R \cap B| &= 80. \end{aligned}$$

It should be clear from the Venn diagram that, by the inclusion-exclusion principle,

$$\begin{aligned} |R \cup B| &= |R| + |B| - |R \cap B|, \\ &= 150 + 105 - 80, \\ &= 175. \end{aligned}$$

It should also be clear from the Venn diagram that

$$\begin{aligned} |N| &= |J| - |R \cup B| \\ &= 300 - 175 \\ &= 125 \end{aligned}$$

since N is the set of all jars that are neither in set R nor in set B.

Complementation rule

This is a variant of the subtraction rule. Suppose that you have a large set S where $|S| = n$ and a subset A of S. Your goal is to determine the size of $|A|$.
Sometimes, it is easier to count the members of S that are not in A. If m members are not in A, then $|A| = n - m$.

Example. Assume there are 26 letters. How many strings of 3 letters are there that contain at least one occurrence of letter z?

Answer. Let S be the set of all 3-letter strings. There are $26^3 = 17576$ of those. Let A be the set of 3-letter strings that contain at least one occurrence of z. We need to find $|A|$.

Let B be the set of 3-letter strings that do not contain z. That just limits you to 25 letters. So there are $25^3 = 15625$ of those.

So $|A| = 17576 - 15625 = 1951$ 3-letter strings that contain z.

Why was it easier to count strings that do not contain z than to count strings that do contain z. There is only one way for a string not to contain a z. But doing a direct way to count strings that do contain z appears to require looking separately at strings with one z, then strings with 2 z's, then strings with 3 z's.

Work homework exercises 6, 14 in exercise set 4.
For problem 14, notice that there are four kinds of positive integers not exceeding n:

1. **Group A: integers that are divible by p;**
2. **Group B: integers that are divible by q;**
3. **Group C: integers that are divible by both p and q;**
4. **Group D: integers that are divible by neither p nor q.**

The problem asks you to determine the size of group D. Start by looking at a small example, where $p = 3$, $q = 5$ and $n = 15$. Show each of the four groups above for this example. How can you determine the size of group A? What about group B? How many members does group C have? How can you derive the size of group D if you know the sizes of groups A, B and C?
Now solve the general problem.