Computer Science 2405
March 25, 2020

Happy Wednesday, March 25.

Today, we continue looking at basics of counting.

Please write up solutions to the problems mentioned in this page and email your solutions to me by the end of the day Friday, March 27. You can either write the solutions in the body of the email or you can attach a separate Microsoft Word document.

Please use subject **2405 hw 4-2**.

Finishing basic counting rules

Read Rosen section 6.1.5 (The Division Rule).

Suppose that you want to count people at a concert, but you only have a way to count legs. The number of people is half the number of legs.

> **Division rule**
> Suppose you have n things that are grouped into groups of size d. Then the number of groups is n/d.

Example. There are $n!$ orders in which n people can sit in a row. For example, if $n = 3$, there are $3! = 6$ orders for A, B and C to sit: ABC, ACB, BAC, BCA, CAB and CBA.

But now suppose that n people sit around a circular table, and there is no designated start position. That means, for $n = 3$, that orders ABC, CAB and BAC are the same. (Write them around a circle to see that they are rotations of one another.)

How many circular orderings are there of n people?

Answer. Since any of the three people can equally be the starting point, there linear orders are in groups of size 3 that are equivalent circular orders. For general n, each group of equivalent circular orders has size n. By the division rule, there are $n!/n = (n-1)!$ different circular orders. So there are $4! = 24$ different ways to sit 5 people around a circular table.

Example. How many positive integer from 1 to 54 are divisible by 9?

Answer. Every 9-th value is divisible by 9. So there are $54/9 = 6$ integers from 1 to 54 that are divisible by 9.

Exercises

Do exercises 7(a-c) and 8(a-d).

Permutations

The basic counting rules feed well into the topics of permutations and combinations, so for now we will skip over section 6.2 (The Pigeonhole Principle) and come back to it.

A *permutation* is an ordering. There are 2 permutations of A and B, namely AB and BA. There are 3 permutations of ABC, namely ABC, ACB, BAC, BCA, CAB and CBA.

We can use the product rule to count permutation. We can choose a permutation by first choosing the first value, then the second value, then the third value, etc.

To count permutations of ABC, there are 3 ways to select the first value (either A, B or C). After selecting the first value, that value is not available for selecting again, so there are only 2 ways to choose the second value. For example, suppose that you decide to start a permutation with A. There are only two available choices for the second value in the permutation, namely B and C. After choosing the second value, there is only one choice of third value.

Clearly, when counting permutations of n things, there are

1. n ways to choose the first value,
2. $n - 1$ ways to choose the second value,
3. $n - 2$ ways to choose the third value,
4. and so on, until there is only one way to choose the last value.

<u>Permutation rule</u>
There are $n! = n(n-1)(n-2)\cdots 1$ permutations of n different values.

The number of permutations of n different things is also called $P(n)$. So $P(n) = n!$.

Exercises

Read Rosen sections 6.3.1 (Introduction) and and 6.3.2 (Permutations). Do exercises 3 and 10(a-d) of homework assignment 4.

r-Permutations

How many ways are there to select 3 things out of 4, if the order in which things are chosen matters, and you are not allowed to choose anything twice? (That is, there are no repetitions.) Using A, B, C and D for the 4 things, the possible orders are:

```
ABC
ABD
ACB
ACD
ADB
ADC
BAC
BAD
BCA
BCD
BDA
BDC
CAB
CAD
CBA
CBD
CDA
CDB
DAB
DAC
DBA
DBC
DCA
DCB
```

There are 4 ways to choose the first value. For each of those choices, there are 3 ways to choose the second value. Then, for each of the choices of the first two values, there are 2 ways to choose the third value.

A selection of r things out of a set S where order matters is called an *r-permutation* of S.

r-Permutation rule
Suppose that n is a positive integer and r is an integer where $0 \leq r \leq n$. There are

$$n(n-1)(n-2)\cdots(n-r+1)$$

r-permutations of a set of size n.

The number of r-permutations of a set of size n is called $P(n, r)$. So

$$\begin{aligned} P(n,r) &= n(n-1)(n-2)\cdots(n-r+1) \\ &= n!/(n-r)! \end{aligned}$$

Example. How many ways are there to select 2 things out of 4 different things if repetitions are not allowed and order matters?

Answer. There are 4 ways to make the first choice and 3 ways to make the second choice. By the product rule, there are $(4)(3) = 12$ ways to do it. Write them all down to make sure that you understand.

Example. How many ways are there to select 2 things out of 4 if order matters but repetitions are allowed?

Answer. This is a more direct application of the product rule. You have 4 choices for the first value and 4 choices for the second value, so there are $(4)(4) = 16$ ways to choose the two values. Write them all down.

Example. How many ways are there to select 5 things out of 7 values if repetitions are not allowed and order matters?

Answer. By the r-permutation rule, there are $(7)(6)(5)(4)(3) = 2520$ ways to do that. (Notice that 5 numbers are multiplied together because we need to make 5 choices. The numbers decrease because repetitions are not allowed.)

Example. How many ways are there to select 5 things out of 7 values if repetitions *are* allowed and order matters?

Answer. There are 7 ways to make each choice, so there are $7^5 = 16807$ ways to make 5 choices. The numbers being multiplied do not decrease because repetitions are allowed.

Exercises

Do exercises 8(e), 9(b).

Combinations

Now we turn to the case of selecting things from a set where order does not matter.

Suppose that n is a positive integer and k is an integer where $0 \leq k \leq n$. Let S be a set of n values. A *k-combination* of S is a subset of S of size k. $C(n, k)$ is defined to be the number of k-combinations that a set of size n has.

For example, suppose $n = 4$, $k = 2$ and $S = \{a, b, c, d\}$. There are 6 2-combinations of S, namely:

```
{a, b}
{a, c}
{a, d}
{b, c}
{b, d}
{c, d}
```

So $C(4, 2) = 6$.

It is easy to derive a formula for $C(n, k)$ by combining the r-permuation rule with the division rule. Notice that there are 12 2-permutations of $\{a, b, c, d\}$, namely:

```
ab
ba
ac
ca
ad
da
bc
cb
bd
db
cd
dc
```

Each 2-combination corresponds to two 2-permutations because there are $2! = 2$ permutations of 2 things.

In general, there are $P(n, k)$ ways to select k things out of n if order matters. Each k-combination corresponds to a group of $k!$ k-permutations.

Combination rule
$C(n, k) = P(n, k)/k!$.

Example. Suppose there are 8 players on a basketball team. How many ways are there to select 5 players to start a game?

Answer. There are C(8,5) = P(8,5)/5! ways to choose 5 players to start. Since

$$\begin{aligned} P(8, 5) &= (8)(7)(6)(5)(4) \\ &= 6720 \end{aligned}$$

and $5! = 120$, there are $6720/120 = 56$ different ways to select 5 players out of 8 to start the game.

You can ease the arithmetic some. Notice that

$$\begin{aligned} C(n, k) &= P(n, k)/k! \\ &= \frac{n!}{(n-k)!k!} \end{aligned}$$

So

$$\begin{aligned} C(8, 5) &= \frac{(8)(7)(6)(5)(4)}{(5)(4)(3)(2)} \\ &= \frac{(8)(7)(6)}{(3)(2)} \\ &= (8)(7) \\ &= 56 \end{aligned}$$

Notation

Combinations crop up all over mathematics. Mathematicians commonly use notation $\binom{n}{k}$ to mean $C(n, k)$, so that they don't need to remind the reader what $C(n, k)$ means. Notice that there is no horizontal line in $\binom{n}{k}$. Notation $\binom{n}{k}$ is read "n choose k."

Exercises

Read Rosen sections 6.3.3 (Combinations). Do exercises 25 and 31(a,b) of homework set 4.