Computer Science 2405
March 27, 2020

Happy Friday, March 27.

Be sure to turn in the homework. You will need to have experience answering questions about the topics that we have been covering.

Submit the answers to today's questions by the end of Monday, March 30.

Today, we will look at

1. Solving sample problems that involve combinations.
2. Some identities involving binomial coefficients.

Rosen section 6.3.3 (Combinations) covers combinations and the identity

$$\binom{n}{r} = \binom{n}{n-r}$$

Rosen section 6.4.2 (Pascal's Identy and Triangle) covers Pascal's identity.

We will skip over 6.4.1 (The Binomial Theorem) and come back to it later because Pascal's identity fits in better with what we are doing here.

Problems involving combinations

Example. How many bit strings are there of length six have exactly three 0's and three 1's?

Answer. You can choose a bit string with three 0's and three 1's by selecting three positions to put 0's. The remaining positions must hold 1. There are $\binom{6}{3}$ ways to select three out of six positions. So the answer is

$$\binom{6}{3} = \frac{(6)(5)(4)}{(3)(2)} = 20.$$

Example. A coin is flipped 5 times and the sequence of results is written down as a string of H's and T's. For example, one possible outcome is $HTTHT$. How many possible outcomes are there that have exactly two heads?

Answer. There are $\binom{5}{2}$ ways to select two positions where there is an H, so there are $\binom{5}{2} = 10$ possible outcomes with exactly two H's.

Example. Again, a coin is flipped 5 times and the sequence of results is written down as a string of H's and T's. How many possible outcomes have at least two H's?

Answer. You can add up the counts for 2, 3, 4 and 5 H's.

$$\begin{aligned}\binom{5}{2} + \binom{5}{3} + \binom{5}{4} + \binom{5}{5} &= 10 + 10 + 5 + 1 \\ &= 26.\end{aligned}$$

Alternatively, you can count the number of outcomes that have 0 and 1 H, and subtract from the total number of outcomes.

$$\begin{aligned}2^5 - \binom{5}{0} - \binom{5}{1} &= 32 - 1 - 5 \\ &= 26.\end{aligned}$$

Exercises

Do exercises 9(c), 26(a-d), 28(a-d), 31(a, b), 33.

An elementary identity

Theorem. Suppose n and r are positive integers where $r \leq n$. Then

$$\binom{n}{r} = \binom{n}{n-r}.$$

Proof 1. We know that

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}.$$

Replacing r by $n-r$ gives

$$\begin{aligned}\binom{n}{r} &= \frac{n!}{(n-r)!(n-(n-r))!} \\ &= \frac{n!}{(n-r)!(r)!}.\end{aligned}$$

Proof 2. We can prove the above theorem without using a formula for $\binom{n}{r}$. Think of the process of choosing r values as painting those r values red. Having done that, paint the remaining (unselected) values blue. For every red subset of size r, there is a corresponding blue subset of size $n-r$.

If there is a one-to-one correspondence (a bijection) between two sets, then those sets must have the same size. So there are the same number of red subsets as blue subsets.

There are $\binom{n}{r}$ ways to choose a red subset of size r values, and $\binom{n}{n-r}$ ways to choose the corresponding blue subset (of size $n-r$). So

$$\binom{n}{r} = \binom{n}{n-r}.$$

Pascal's identity

Rosen section 6.4.2 (Pascal's Identity and Triangle) covers Pascal's identity, which is as follows.

Theorem. Suppose n and k are positive integers with $k \leq n$. Then

$$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}.$$

Proof. Think about how you might choose k different values from a set $S = \{x_1, x_2,, \ldots, x_{n+1}\}$. You have a choice of selecting x_{n+1} or not selecting it.

1. If you select x_{n+1}, then you must make $k-1$ remaining choices out of set $\{x_1, x_2,, \ldots, x_n\}$, and there are $\binom{n}{k-1}$ ways to do that.

2. If you do not select x_{n+1}, then you must make k remaining choices out of set $\{x_1, x_2,, \ldots, x_n\}$, and there are $\binom{n}{k}$ ways to do that.

The sum rule tells us that there are a total of $\binom{n}{k-1} + \binom{n}{k}$ ways to do either (1) or (2). (Notice that there is no overlap.) So

$$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}.$$

Pascal's triangle

Pascal's identity can be used as a way to compute binomial coefficients that does not involve factorials or even multiplication. First, write the following two rows.

```
    1
  1   1
```

The value in the first row is $\binom{0}{0}$. There is exactly 1 way to choose nothing from an empty set. The next row contains $\binom{1}{0}$ and $\binom{1}{1}$. There is exactly 1 way to choose no things from set $\{a\}$ and exactly 1 way to choose one thing from set $\{a\}$. So the second row is

$$\binom{1}{0} \quad \binom{1}{1}$$

Now add a row containing three values,

$$\binom{2}{0} \quad \binom{2}{1} \quad \binom{2}{2}$$

```
      1
    1   1
  1   2   1
```

Notice that $2 = 1 + 1$ is the sum of the two values immediately above it to the left and the right. Pascal's identity tells us that

$$\binom{2}{1} = \binom{1}{0} + \binom{1}{1}.$$

Now add another row for $n = 3$ by writing a 1 at each end and filling in the sums of the values just above to the left and right.

```
        1
      1   1
    1   2   1
  1   3   3   1
```

The bottom row contains

$$\binom{3}{0} \quad \binom{3}{1} \quad \binom{3}{2} \quad \binom{3}{3}$$

and you can just read them off. Continuing, Pascal's triangle goes as follows.

```
          1
        1   1
      1   2   1
    1   3   3   1
  1   4   6   4   1
1   5   10  10  5   1
```

Pascal's triangle keeps going forever. There is a larger part of it in Rosen, section 6.4.2. To find $\binom{5}{3}$, count in the row whose second number is 5, starting the count at 0.

```
1   5   10  10  5   1
0   1    2   3
```

You get $\binom{5}{3} = 10$, the fourth value in the fifth row.

Exercises

Do problem 18 from homework set 4.