Computer Science 2405
April 1, 2020

Happy Wednesday, April 1.

We will end material from chapter 6 with The Pigeonhole Principle.

The Pigeonhole Principle

The Pigeonhole Principle is simple and obvious. If you have 10 boxes and you are only allowed to put one thing in each box, you cannot put more than 10 things into those boxes. Here it is stated in a slightly different way.

> **The Pigeonhole Principle**
> If you put $n + 1$ things into n boxes, then there must end up with a box that has two or more things.

Although it is simple, The Pigeonhole principle is remarkably useful.

Example. Show that, among 32 people, there must be two who were born on the same day of the month.

Answer. There are only 31 different days of the month. 32 people cannot possibly all have birthdays on different days of the month.

Example. A bowl contains 20 red balls and 20 blue balls. Suppose that you select 21 balls from the bowl. Argue that you must have selected at least one red ball.

Answer. There are only 20 nonred balls.

Example. Suppose that you select 10 ordered pairs of integers. Show that, among those selected ordered pairs, there are two, (a_1, b_1), (a_2, b_2), such that $a_1 \equiv a_2 \pmod 3$ and $b_1 \equiv b_2 \pmod 3$.

Answer. The only thing that matters about each integer is its remainder when you divide it by 3. So we can assume that only integers 0, 1 and 2 can be selected, and we are asking for $a_1 = a_2$ and $b_1 = b_2$.

There are $3 \times 3 = 9$ different ordered pairs (x, y) where x and y are in $\{0, 1, 2\}$. Since 10 ordered pairs have been selected, two of them must be identical.

Generalized Pigeonhole Principle

There is a more general version of The Pigeonhole Principle that is also simple and obvious.

> **Generalized Pigeonhole Principle**
> Suppose $k > 0$. If you put n things into k boxes, then there must be a box that has at least $\lceil n/k \rceil$ things in it.

Proof. If there are at most m things per box, and there are k boxes, you can only have at most km total things. That is, if you actually have n things, then

$$n \leq km.$$

So

$$n/k \leq m.$$

Since m has to be an integer, it must be the case that

$$m \geq \lceil n/k \rceil.$$

For example, if $n/k = 4.5$ then $m \geq 4.5$, which means $m \geq 5$ because m is an integer.

Example. How many cards must be selected from a deck of playing cards to ensure that, among the selected cards, there must be at least 3 of the same suit?

Answer. There are 4 suits in a deck. They are the boxes. If you select 9 cards, then there must be a box that has $\lceil 9/4 \rceil = 3$ cards selected.

Exercises

Read Rosen section 6.2.1 and 6.2.2.

Do exercises 13, 16, 17, 20, 22, 23, 24, 31(c) from homework set 4. Not all of those involve the Pigeonhole Principle, but several do.

There are many more difficult applications of The Pigeonhole Principle, some discussed in Rosen section 6.2.3, but each one has a particular trick associated with it, so they are not of a great deal of interest to us. So let's stop now.