Computer Science 2405
April 8, 2020

Happy Wednesday, April 8.

There will be an exam on Monday, April 13. I will post the exam on the course web page at 11:00. Please send me a Microsoft Word document with your answers, as an attachment in an email, by 1:30. (I will give you an extra half hour.) The name of the attachment should begin with your last name followed by your first name. Do not send me a file called "Exam 4.docx".

If you need to attach graphics files, give them names that begin with your last name, then your first name.

Breaking down the problem of solving recurrences

Now we embark on solving recurrences. This material is in Rosen sections 8.2.1 and 8.2.2.

We will only be concerned with *linear* recurrences, which have the general form

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k} + f(n)$$

where c_1, $c_2, \ldots, c_k$ are real numbers and $f(n)$ is some function of n.

The degree of a recurrence

By definition, the *degree* of recurrence

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k} + f(n)$$

is k. For example,

$$a_n = c_1 a_{n-1} + n^2$$

has degree 1 and

$$a_n = c_1 a_{n-1} + c_2 a_{n-2}$$

has degree 2.

Notice that the recurrence is called linear even if it has degree 2. Also, $f(n)$ might be a degree 3 polynomial, but the recurrence is still said to be a linear recurrence. The linearity comes from the fact that you don't see something like a_1^2. A linear recurrence is linear in the sequence values a_i.

Homogeneous recurrences

A linear recurrence is *homogeneous* if it has the form

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}.$$

That is, the $f(n)$ part is 0.

Homogeneous recurrences are easier to solve than inhomogeneous recurrences, so they are a good place to start. Also, the solution to an inhomogenous recurrence involves a solution to the corresponding homogeneous recurrence (where you throw out any $f(n)$ term).

Characteristic equations

Consider homogeneous recurrence

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}.$$

The *characteristic equation* of this recurrence is equation

$$r^k = c_1 r^{k-1} + c_2 r^{k-2} + \cdots + c_k$$

or, equivalently,

$$r^k - c_1 r^{k-1} - c_2 r^{k-2} - \cdots - c_k = 0.$$

For example, recurrence

$$a_n = 2a_{n-1} + 5a_{n-2}$$

has characteristic equation

$$r^2 - 2r - 5 = 0$$

and recurrence

$$a_n = 4a_{n-1} - 2a_{n-2} + a_{n-3}$$

has characteristic equation

$$r^3 - 4r^2 + 2r - 1 = 0.$$

Be careful about missing terms. Recurrence

$$a_n = 2a_{n-1} + a_{n-3}$$

is equivalent to

$$a_n = 2a_{n-1} + 0a_{n-2} + a_{n-3}$$

and its characteristic equation is

$$r^3 - 2r^2 - 1 = 0.$$

General solutions

The *general solution* to a homogeneous recurrence does not depend on the initial conditions. Instead, it contains constants that can be adjusted to any given initial conditions.

Degree 1 homogeneous recurrences

To get an idea where we are headed, let's look at a particular degree 1 recurrence

$$a_n = 3a_{n-1}$$

with initial value $a_0 = 1$. Then

$$\begin{aligned} a_1 &= 3 \\ a_2 &= 3^2 \\ a_3 &= 3^3 \end{aligned}$$

and clearly $a_n = 3^n$. Based on this example, it should not come as a surprise that exponential functions like 2^n and 3^n crop up in solutions of linear recurrences.

The characteristic equation of recurrence

$$a_n = 3a_{n-1}$$

is

$$r = 3.$$

The solution for r tells you the base of the exponential function.

What if the initial value is $a_0 = 2$? Then

$$\begin{aligned} a_1 &= 2(3) \\ a_2 &= 2(3^2) \\ a_3 &= 2(3^3) \end{aligned}$$

Generalizing what we see here:

Solution to a linear homogeneous equation of degree 1

Recurrence

$$a_n = c_1 a_{n-1}$$

has a general solution $a_n = d(c_1)^n$. That is:

(a) For any given value of a_0, there is a constant d so that $a_n = d(c_1)^n$.

(b) Every solution to this recurrence (for any initial value) has the form $a_n = d(c_1)^n$ for some constant d.

(It turns out that $d = a_0$. But it is helpful to see this simple solution in a form that looks similar to the solution form for recurrences of degree greater than 1, and they don't have such an easy way to determine the constants.)

Degree 2 homogeneous recurrences without repeated roots

To find a general solution of a degree 2 homogeneous linear recurrence, start by finding all solutions of the characteristic equation. Here is a recipe that works when the characteristic equation has two different real-valued solutions.

Linear homogeneneous equation of degree 2 with two roots

Consider recurrence

$$a_n = c_1 a_{n-1} + c_2 a_{n-2}.$$

Its characteristic equation is

$$r^2 - c_1 r - c_2 = 0.$$

Suppose that r_1 and r_2 are real solutions of the characteristic equation where $r_1 \neq r_2$. Then

(a) For any given values of a_0 and a_1, there exist constants d_1 and d_2 so that $a_n = d_1 r_1^n + d_2 r_2^n$.

(b) Every solution to this recurrence (regardless of the initial values) has the form $a_n = d_1 r_1^n + d_2 r_2^n$ for some constants d_1 and d_2.

Example. Find the general solution of recurrence

$$a_n = 5a_{n-1} - 6a_{n-2}.$$

Answer. The characteristic equation is

$$r^2 - 5r + 6 = 0$$

with solutions $r_1 = 2$ and $r_2 = 3$. The general solution is

$$a_n = d_1 2^n + d_2 3^n.$$

Example. Find the solution of recurrence

$$a_n = 5a_{n-1} - 6a_{n-2}$$

with initial values $a_0 = 1$ and $a_1 = 2$.

Answer. The first few values after a_1 are

$$\begin{aligned} a_2 &= 5a_1 - 6a_0 \\ &= 5(2) - 6(1) \\ &= 4 \\ a_3 &= 5a_2 - 6a_1 \\ &= 5(4) - 6(2) \\ &= 8 \\ a_4 &= 5a_3 - 6a_2 \\ &= 5(8) - 6(4) \\ &= 16 \end{aligned}$$

We have already seen that the general solution is $a_n = d_1 2^n + d_2 3^n$ where constants d_1 and d_2 need to be adjusted to the initial conditions. We can find d_1 and d_2 by substituting $n = 1$ and $n = 2$ into the general solution.

$$\begin{aligned} 1 &= a_0 \\ &= d_1 2^0 + d_2 3^0 \\ &= d_1 + d_2 \\ 2 &= a_1 \\ &= d_1 2^1 + d_2 3^1 \\ &= 2d_1 + 3d_2 \end{aligned}$$

Now we can find the values of d_1 and d_2 by solving those two linear equations in two unknowns. The first equations tells us that

$$d_2 = 1 - d_1.$$

Substituting that into the second equation gives

$$2 = 2d_1 + 3(1 - d_1)$$

or $d_1 = 1$ and $d_2 = 1 - d_1 = 0$. That gives a solution of $a_n = 2^n$.

Exercises

1. Solve recurrence
$$a_n = 5a_{n-1} - 6a_{n-2}$$
with initial values $a_1 = 1$ and $a_2 = 3$.

2. Solve recurrence
$$a_n = 5a_{n-1} - 6a_{n-2}$$
with initial values $a_1 = 1$ and $a_2 = 4$.

Degree 2 homogeneous recurrences with repeated roots

Now we turn to the case where the characteristic equation has only one real-valued solution.

Linear homogeneneous equation of degree 2 with one root

Consider recurrence

$$a_n = c_1 a_{n-1} + c_2 a_{n-2}.$$

Its characteristic equation is

$$r^2 - c_1 r - c_2 = 0.$$

Suppose that the characteristic equation has only one solution, r_1.

(a) For any givens value of a_0 and a_1, there exist constants d_1 and d_2 so that $a_n = d_1 r_1^n + d_2 n r_1^n$.

(b) Every solution to this recurrence (regardless of the initial values) has the form $a_n = d_1 r_1^n + d_2 n r_1^n$ for some constants d_1 and d_2.

Example. Find the general solution to recurrence

$$a_n = 4a_{n-1} - 4a_{n-2}.$$

Answer. The characteristic equation is

$$r^2 - 4r + 4 = 0.$$

Since $r^2 - 4r + 4 = (r-2)^2$, there is only one solution, $r = 2$. (We say that 2 is a solution with *multiplicity* 2.)

The general solution is

$$a_n = d_1 2^n + d_2 n 2^n.$$

Example. Find the solution to the preceding recurrence with initial values $a_0 = 0$ and $a_1 = 2$.

Answer. Plugging in $n = 0$ into the general solution gives

$$0 = d_1(2^0) + d_2(0)(2^0)$$

so $d_1 = 0$. Plugging in $n = 1$ gives

$$2 = d_1(2^1) + d_2(1)(2^1) = 2d_2$$

so $d_2 = 1$. That means the solution is $a_n = n2^n$. Let's try it for a couple of values. By the recurrence,

$$\begin{aligned} a_2 &= 4a_1 - 4a_0 \\ &= 8 \end{aligned}$$

The formula says that

$$\begin{aligned} a_2 &= (2)(2^2) \\ &= 8 \end{aligned}$$

By the recurrence,

$$\begin{aligned} a_3 &= 4a_2 - 4a_1 \\ &= 4(8) - 4(2) \\ &= 24 \end{aligned}$$

The formula says that

$$\begin{aligned} a_3 &= (3)(2^3) \\ &= 24 \end{aligned}$$

Exercises

Do exercises 8 and 9 (all parts) from homework set 5. Submit it for checking by the end of Friday, April 17.