Computer Science 2405
April 15, 2020

Happy Wednesday, April 15.

Higher degree recurrences

We have seen rules for solving linear homogeneous recurrences of degree 1 and 2. Those rules generalize to higher degrees, and the generalizations are in Theorems 3 and 4 in Rosen, Section 8.2.2. However, we will restrict ourselves to solving recurrences of degree no more than 2.

Nonhomogeneous recurrences

A nonhomogeneous linear recurrence with constant coefficients has the general form

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k} + F(n)$$

where c_1, c_2, ... , c_n are real numbers and $F(n)$ is some function of n. Examples are

$$a_n = a_{n-1} + n$$

and

$$a_n = 2a_{n-1} - a_{n-2} + 2^n.$$

Associated homogeneous recurrences

Every nonhomogeneous recurrence has an associated homogeneous recurrence that is obtained by replacing the $F(n)$ term by 0. For example, the associated homogeneous recurrence of

$$a_n = a_{n-1} + n$$

is

$$a_n = a_{n-1}$$

and the associated homogeneous recurrence of

$$a_n = 2a_{n-1} - a_{n-2} + 2^n$$

is

$$a_n = 2a_{n-1} - a_{n-2}.$$

Particular solutions

A *particular solution* of a nonhomogeneous recurrence is a solution for particular initial values.

Example. Let's look at recurrence

$$a_n = a_{n-1} + n$$

with initial value $a_0 = 0$. Here are the first few values of sequence $\{a_n\}$ (a notation that means $(a_0, a_1, a_2, \ldots)$).

$$\begin{aligned}
a_0 &= 0 \\
a_1 &= a_0 + 1 \\
&= 0 + 1 \\
&= 1 \\
a_2 &= a_1 + 2 \\
&= 1 + 2 \\
&= 3 \\
a_3 &= a_2 + 3 \\
&= 3 + 3 \\
&= 6 \\
a_4 &= a_3 + 4 \\
&= 6 + 4 \\
&= 10
\end{aligned}$$

It is easy to see that a_n is just the sum of the first n positive integers. A closed-form solution of recurrence

$$a_n = a_{n-1} + n$$

with initial value $a_0 = 0$ is

$$a_n = \frac{n(n+1)}{2}$$

and we call that a particular solution to recurrence

$$a_n = a_{n-1} + n.$$

In fact, it is easy to see that $a_n = f(n)$ where $f(n) = n(n+1)/2$ is a solution of recurrence $a_n = a_{n-1} + n$ by substituting it into the recurrence. Replacing a_n by $f(n)$ and replacing a_{n-1} by $f(n-1)$ in the recurrence gives

$$n(n+1)/2 = (n-1)n/2 + n$$

which is easily seen to be true.

If we choose a different initial value, we get a different sequence. For example, if $a_0 = 1$, then the sequence is

$$\begin{aligned}
a_0 &= 1 \\
a_1 &= a_0 + 1 \\
&= 1 + 1 \\
&= 2 \\
a_2 &= a_1 + 2 \\
&= 2 + 2 \\
&= 4 \\
a_3 &= a_2 + 3 \\
&= 4 + 3 \\
&= 7 \\
a_4 &= a_3 + 4 \\
&= 7 + 4 \\
&= 11
\end{aligned}$$

The solutions for that sequence is

$$a_n = \frac{n(n-1)+2}{2}.$$

Example. Consider recurrence

$$a_n = 2a_{n-1} - a_{n-2} + 2^n$$

with initial values $a_0 = 0$ and $a_1 = 0$. Here are the first few values in sequence $\{a_n\}$.

$$\begin{aligned}
a_0 &= 0 \\
a_1 &= 0 \\
a_2 &= 2a_1 - a_0 + 2^2 \\
&= 0 - 0 + 4 \\
&= 4 \\
a_3 &= 2a_2 - a_1 + 2^3
\end{aligned}$$

$$
\begin{aligned}
&= 8 - 0 + 8 \\
&= 16 \\
a_4 &= 2a_3 - a_2 + 2^4 \\
&= 32 - 4 + 16 \\
&= 44
\end{aligned}
$$

A closed-form solution is $a_n = f(n)$ where $f(n) = 4(2^n - n - 1)$. You can check that $f(0) = 0$ and $f(1) = 0$, so $f(n)$ gets the initial values right. Plugging $a_n = f(n)$ into the recurrence gives

$$f(n) = 2f(n-1) - f(n-2) + 2^n$$

or

$$4(2^n - n - 1) = 2(4(2^{n-1} - (n-1) - 1)) - 4(2^{n-2} - (n-2) - 1) + 2^n$$

which we need to show is true. Multiplying out the right hand side gives

$$4(2^n - n - 1) = 4(2^n) - 8(n-1) - 8 - 2^n + 4(n-2) + 4 + 2^n$$

which you should be able to check is true.

We say that $a_n = 4(2^n - n - 1)$ is a particular solution of recurrence

$$a_n = 2a_{n-1} - a_{n-2} + 2^n.$$

Difference between particular solutions

Let's start with a nonhomogenous recurrence

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k} + F(n).$$

We can find two different sequences $\{a_n\}$ and $\{b_n\}$, *with different initial values*, both satisfying that recurrence. That is,

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k} + F(n)$$

and

$$b_n = c_1 b_{n-1} + c_2 b_{n-2} + \cdots + c_k b_{n-k} + F(n)$$

for $n \geq 2$. Subtracting the second equation from the first gives

$$a_n - b_n = c_1(a_{n-1} - b_{n-1}) + c_2(a_{n-2} - b_{n-2}) + \cdots + c_k(a_{n-k} - b_{n-k}).$$

If we define $d_n = a_n - b_n$, then

$$d_n = c_1 d_{n-1} + c_2 d_{n-2} + \cdots + c_k d_{n-k}.$$

So d_n must be a solution to the associated homogeneous recurrence. That tells us that

$$a_n = b_n + d_n$$

and particular solution for a_n can be found by taking a different particular solution for b_n and adding a solution of the associated homogeneous recurrence.

Example. We have seen two particular solutions to recurrence

$$a_n = a_{n-1} + n$$

namely $a_n = n(n-1)/2$ when $a_0 = 0$ and $a_n = n(n-1)/2 + 1$ when $a_0 = 1$. They differ by a constant that is added to one of them but not the other. The associated homogeneous equation

$$a_n = a_{n-1}$$

has solution $a_n = a_0$. That is, a_n is a constant.

General rule relating particular solutions

The following is Theorem 5 in section 8.2.2 of Rosen.

> Suppose that $f(n)$ is a particular solution of recurrence
>
> $$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k} + F(n).$$
>
> That is, $a_n = f(n)$ for all $n \geq 0$ for certain initial values. Then every particular solution $g(n)$ of that recurrence has the form $g(n) = f(n) + h(n)$ where $h(n)$ is a solution of homogenous recurrence
>
> $$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}.$$

Exercises

Do exercises 1, 10 and 12 in homework set 5. Submit your answers by email by Monday, April 20. Please refer to it as assignment 5-3. Attach file(s) whose names begin with your last name followed by your first name.