Computer Science 2405
April 17, 2020

Happy Friday, April 17.

We only have two subjects left. Today, we look at how to find a particular solution to an nonhomogeneous recurrence of a restricted form. After that we will only have section 8.3 on divide-and-conquer recurrences left.

Solving nonhomogeneous linear recurrences of a restricted form

The following is Theorem 6 in section 8.2.3 of Rosen.

Theorem 6. A formula for finding a solution to a nonhomogenous recurrence

Suppose that $F(n)$ is a function of n that has the form of a polynomial in n times a function of the form s^n where s is a constant. Suppose

$$F(n) = (b_t n^t + b_{t-1} n^{t-1} + \cdots + b_1 n + b_0)(s^n).$$

Let R be the following nonhomogenous reccurrence.

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k an - k + F(n).$$

The characteristic equation of R is

$$r^n - c_1 r^{n-1} - c_2 r^{n-2} - \cdots - c_k r^{n-k} = 0.$$

The homogeneous recurrence R_h associated with R is:

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k an - k.$$

Suppose that $a_n = h(n)$ is a general solution of R_h.

Case 1. Suppose that $r = s$ is not a solution of the above characteristic equation. Then, for any given initial conditions, recurrence R has a solution of the form

$$a_n = (p_t n^t + p_{t-1} n^{t-1} + \cdots p_1 n + p_0)(s^n) + h(n)$$

where $p_0, p_2, \ldots, p_t$ are constants that depend on the initial conditions.

Case 2. Suppose that $r = s$ is a solution of the above characteristic equation with multiplicity m. (That is, $(r-s)^m$ is a factor of $r^n - c_1 r^{n-1} - c_2 r^{n-2} - \cdots - c_k r^{n-k}$ but $(r-s)^{m+1}$ is not a factor of $r^n - c_1 r^{n-1} - c_2 r^{n-2} - \cdots - c_k r^{n-k}$.)

Then, for any given initial conditions, recurrence R has a solution of the form

$$a_n = n^m (p_t n^t + p_{t-1} n^{t-1} + \cdots p_1 n + p_0)(s^n) + h(n)$$

where $p_0, p_2, \ldots, p_t$ are constants that depend on the initial conditions.

Example. Find a closed form solution of nonhomogenous recurrence R:

$$a_n = 3a_{n-1} + n \tag{1}$$

where $a_0 = 0$.

Answer.

1. The associated homogeneous recurrence R_h is

   $$a_n = 3a_{n-1}.$$

   The characteristic equation is

   $$r - 3 = 0$$

   which has solution $r = 3$ (with multiplicity 1). A general solution to R_h is

   $$a_n = c3^n$$

   where c is a constant.

2. Using the notation of Theorem 6, function $F(n) = n$ is the same as $F(n) = (1n+0)(1^n)$, so $s = 1$, $t = 1$, $p_1 = 1$ and $p_0 = 0$. Since s is not a solution of the characteristic equation, there must be a solution of nonhomogenous recurrence R of the form

   $$a_n = (p_1 n + p_0)(1^n) + c3^n \tag{2}$$

   where p_1, p_0 and c are constants that depend on the initial conditions.

3. Now it is a matter of determining the values of constants p_1, p_0 and c. Making direct use of recurrence (1):

   $$\begin{aligned} a_0 &= 0 \\ a_1 &= 3a_0 + 1 \\ &= 1 \\ a_2 &= 3a_1 + 2 \\ &= 3 + 2 \\ &= 5 \\ a_3 &= 3a_2 + 3 \\ &= 15 + 3 \\ &= 18 \end{aligned}$$

Since there are three constants to determine, we need three equations. Using solution (2) for $n = 0$, 1 and 2 yields the following equations.

$$\begin{aligned} 0 = a_0 &= (p_1(0) + p_0)(1^0) + c3^0 \\ 1 = a_1 &= (p_1(1) + p_0)(1^1) + c3^1 \\ 5 = a_2 &= (p_1(2) + p_0)(1^2) + c3^2 \end{aligned}$$

After simplifying those three equations, we have:

$$\begin{aligned} 0 &= p_0 + c \\ 1 &= p_1 + p_0 + 3c \\ 5 &= 2p_1 + p_0 + 9c \end{aligned}$$

From the first of those equations, $p_0 = -c$. Substituting that into the second and third equations gives

$$\begin{aligned} 1 &= p_1 + 2c \\ 5 &= 2p_1 + 8c \end{aligned}$$

The first of those two equations indicates that $p_1 = 1 - 2c$. Substituting that into the last equation gives

$$5 = 2(1 - 2c) + 8c$$

or

$$c = 3/4.$$

Substituting the value of c into the formulas for p_0 and p_1 gives $p_1 = -1/2$ and $p_0 = -3/4$.

4. Now it is just a matter of substituting the known values of c, p_0 and p_1 into (2).

$$\begin{aligned} a_n &= (p_1 n + p_0)(1^n) + c3^n \\ &= -n/2 - 3 + (3/4)3^n \\ &= \frac{3^{n+1} - 2n - 3}{4} \end{aligned}$$

5. Let's check that. Replacing n by 0, 1 and 2 gives

$$\begin{aligned} a_0 &= \frac{3^{0+1} - 2(0) - 3}{4} \\ &= 0 \end{aligned}$$

$$
\begin{aligned}
a_1 &= \frac{3^{1+1} - 2(1) - 3}{4} \\
&= \frac{9 - 2 - 3}{4} \\
&= 1 \\
a_2 &= \frac{3^{2+1} - 2(2) - 3}{4} \\
&= \frac{27 - 4 - 3}{4} \\
&= 5
\end{aligned}
$$

It should come as no surprise our formula works for $n = 0$, 1 and 2. The values of c, p_0 and p_1 were chosen expressly to make it work out for $n = 0$, 1 and 2. The real test comes for values of n that are larger than 2. Notice that, according to our solution,

$$
\begin{aligned}
a_3 &= \frac{3^{3+1} - 2(3) - 3}{4} \\
&= \frac{81 - 6 - 3}{4} \\
&= 18
\end{aligned}
$$

so our solution at least works for $n = 3$.

Exercises

Note: exercise 12 was assigned in the previous lecture in error. You were not ready for it. But you are ready now. Do exercises 11 and 12 in homework set 5. Submit by Wednesday, April 22.