Computer Science 4602
Fall 2020
Quiz 1

You have 30 minutes. Answer all of the questions. You may use one prepared 8.5×11 sheet of paper during the exam. **Check your work**.

1. True/False. For the first-order formulas, the universe of discourse is the set of all integers.

   (a) Every alphabet is finite.
   True, by the definition of an alphabet.

   (b) Every alphabet is nonempty.
   True, by the definition of an alphabet.

   (c) Every language is infinite.
   False.

   (d) Every language is nonempty.
   False. $\{\}$ is a language.

   (e) A string over alphabet Σ can be infinitely long.
   False. By definition, a string is a finite sequence of symbols.

   (f) The empty set is a language.
   True.

   (g) $\{\} \cup S = S$ for every set S.
   True.

   (h) $\{\} \cap S = S$ for every set S.
   False. $\{\} \cap S = \{\}$ for every set S.

   (i) $\{\} \in S$ for every set S.
   False. For example, $\{\} \notin \{1, 2, 3\}$.

   (j) $\{\} \subseteq S$ for every set S.
   True. Every member of $\{\}$ is a member of S (vacuously).

   (k) $S \cap S = \{\}$ for every set S.
   False. $S \cap S = S$ for every set S.

(l) $S - S = \{\}$ for every set S.

True. There are no things that are in S but not in S.

(m) $\forall x \exists y (x^2 = y)$.

True. For a given x, choose $y = x^2$.

(n) $\forall x \exists y (x = y^2)$.

False. This says that all integers are perfect squares.

(o) $\forall x \exists y (x + y = 0)$.

True. For a given x, choose $y = -x$.

2. How many rows are in the truth table of propositional formula $(P \wedge Q) \vee (Q \rightarrow (\neg R \rightarrow S)) \vee T$?

   There are 5 variables. So there are $2^5 = 32$ rows. If there are n variables there are 2^n rows.

3. Prove that, for every real number x, if x is irrational then $1/x$ is irrational.

   Proof 1. We use the contrapositive. Recall that $p \rightarrow q \equiv \neg q \rightarrow \neg p$.

   Choose an arbitrary x. The contrapositive of the claim is: $\neg(1/x \text{ is irrational}) \rightarrow \neg(x \text{ is irrational})$. But saying that a number is not irrational is equivalent to saying that it is rational. So our goal is to show

   Goal: If $1/x$ is rational then x is rational.

   Suppose that $1/x$ is rational. Then there are integers m and n ($n \neq 0$) where $1/x = m/n$. Since $1/x$ can never be 0, we can take reciprocals of both sides, giving $x = n/m$.

   That shows that x is rational provided we know that $m \neq 0$. But $m/n = 1/x$ and $1/x \neq 0$, so $m \neq 0$.

   Proof 2. By contradiction. We are asked to prove $\forall x(x \text{ is irrational} \rightarrow 1/x \text{ is irrational})$. The negation of that is

$$
\begin{aligned}
&\neg(\forall x(x \text{ is irrational} \rightarrow 1/x \text{ is irrational})) \\
&\equiv \exists x(\neg(x \text{ is irrational} \rightarrow 1/x \text{ is irrational}) \\
&\equiv \exists x(x \text{ is irrational} \wedge 1/x \text{ is rational})
\end{aligned}
$$

   since $\neg(p \rightarrow q) \equiv (p \wedge \neg q)$. Our goal is to prove a contradiction.

   Ask someone to provide a value x so that

- x is irrational.
- $1/x$ is rational.

It is more useful here to know that a number is rational than to know that a number is not rational. Since $1/x$ is rational, there must exists m and n ($n \neq 0$) where $1/x = m/n$.

As in the preceding proof, we can conclude that x is rational. But that contradicts the assertion above that x is irrational.

Common mistakes.

- Confusing the words *contrapositive* and *contradiction*.
- Trying a proof by contradiction, but using an incorrect negation of the claim. Take that step carefully. Don't shoot from the hip.
- Trying a proof by contrapositive, but thinking that the contrapositive of (x is irrational $\rightarrow$ $1/x$ is irrational) is (x is rational $\rightarrow$ $1/x$ is rational). Logically, $(\neg p \rightarrow \neg q)$ is called the *inverse* of $(p \rightarrow q)$. Formula $(\neg p \rightarrow \neg q)$ is equivalent to the *converse* $(q \rightarrow p)$ of $(p \rightarrow q)$.
- Trying a direct proof. Assuming that x is irrational does you little good. It tells you that something does not exist. Where do you go from there? It is much easier to work from knowledge that something does exist.