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1 Review of Propositional Logic

This section reviews propositional logic, which you should already have seen.

BIG IDEA: Logic lies at the heart of mathematical reasoning. It is also essential for an understanding of how computers work.

Additional reading: Cummings, Chapter 5.

1.1 Syntax of Propositional Logic

A *propositional formula* is an expression of propositional logic, such as $p \to q$. A propositional formula is also called a compound proposition. I will use the term propositional formula.

The *syntax* of propositional logic only says what a propositional formula looks like. It does not say what a propositional formula means. We use A, B, C and ϕ (Greek letter phi) to name arbitrary propositional formulas.

Definition 1.1. A *propositional formula* is defined as follows.

1. Symbols $\mathbf{T}$ and $\mathbf{F}$ are propositional formulas.
2. A *propositional variable* is a propositional formula. We will use p, q, r and s, possibly with subscripts, as propositional variables and X to refer to an arbitrary variable.
3. If A and B are propositional formulas then so are
   (a) $A \vee B$,
   (b) $A \wedge B$,
   (c) $A \to B$,
   (d) $A \leftrightarrow B$,

(e) $\neg A$,

(f) (A).

For example, each of the following is a propositional formula.

- p
- $p \vee q$
- $p \wedge \neg q$
- $p \wedge q \wedge r$
- $q \vee p \wedge r$
- $(r \wedge \mathbf{T}) \vee \neg q$
- $(p \to (q \to p))$

Operator $\vee$ is read "or", $\wedge$ is read "and", $\to$ is read "implies", $\leftrightarrow$ is read "if and only if" and $\neg$ is read "not".

1.1.1 Precedence and Associativity

Rules of *precedence* and *associativity* determine how you break a propositional formula into subformulas. Higher precedence operators are done first. The following lists operators by precedence, from highest to lowest.

Precedence	
parentheses	high
$\neg$	
$\wedge$	
$\vee$	
$\to$	
$\leftrightarrow$	low

For example, $p \vee q \wedge r$ is understood to have the same structure as $p \vee (q \wedge r)$ since $\wedge$ has higher precedence than $\vee$.

Associativity determines how an expression is broken into subexpressions when it involves two or more occurrences of the same operator. We assume that operators $\vee$ and $\wedge$ are done from left to right. That is, they are *left-associative*. (Associativity is like the wind. A north wind blows from north to south.) For example, $p \vee q \vee r$ has the same structure as $(p \vee q) \vee r$. Associativity does not really matter for $\vee$ and $\wedge$ because they are *associative operators*. That is, $(p \vee q) \vee r$ and $p \vee (q \vee r)$ always have the same value.

Associativity does matter for some operators, so it is wise to think about it. By convention, operators $\rightarrow$ and $\leftrightarrow$ are done from right to left. So $p \rightarrow q \rightarrow r$ has the same meaning as $p \rightarrow (q \rightarrow r)$. However, we will always parenthesize when the associativity of $\rightarrow$ and $\leftrightarrow$ would be needed if the parentheses were omitted, to avoid confusion.

1.2 Meaning of Propositional Logic

The meaning of a propositional formula can only be defined when the values of all of its variables are given. Each variable can be true or false.

Definition 1.2. A *truth-value assignment* is a set of components of the form $X = V$ where X is a variable and V is either T or F. For example, $\{p{=}\text{T},\ q{=}\text{F}\}$ is a truth-value assignment.

Definition 1.3. If a is a truth-value assignment and X is a variable then $a(X)$ is the value (T or F) that a gives for variable X. For example, if a is $\{p{=}\text{T},\ q1{=}\text{F}\}$ then $a(p) = \text{T}$ and $a(q) = \text{F}$.

Definition 1.4. Suppose that ϕ is a propositional formula and a is a truth-value assignment that defines every variable that occurs in ϕ. Notation $(a \dashv \phi)$ indicates the value of ϕ (either T or F) when variables have values given by a. Specifically:

1. $(a \dashv \mathbf{T}) = \text{T}$. That is, symbol **T** is always true.
2. $(a \dashv \mathbf{F}) = \text{F}$. That is, symbol **F** is always false.

3. If X is a variable then $(a \dashv X) = a(X)$. That is, X has the value that it is given by truth-value assignment a.

4. $(a \dashv A \vee B)$ is T if at least one of $(a \dashv A)$ and $(a \dashv B)$ is T, and is F otherwise. For example, $(\{p{=}\mathrm{T},\ q{=}\mathrm{F}\} \dashv p \vee q)$ is T because $(\{p{=}\mathrm{T},\ q{=}\mathrm{F}\} \dashv p)$ is T, and we only need one of p and q to be true.

5. $(a \dashv A \wedge B)$ is T if both of $(a \dashv A)$ and $(a \dashv B)$ are T, and is F otherwise. For example, $(\{p{=}\mathrm{T},\ q{=}\mathrm{F}\} \dashv p \wedge q)$ is F because $(\{p{=}\mathrm{T},\ q{=}\mathrm{F}\} \dashv p)$ and $(\{p{=}\mathrm{T},\ q{=}\mathrm{F}\} \dashv q)$ are not both T.

6. $(a \dashv A \rightarrow B)$ has the same meaning as $(\neg A) \vee B$. Implication is discussed further below.

7. $(a \dashv A \leftrightarrow B)$ is true if $(a \dashv A)$ and $a \dashv B$ have the same value. For example, $(\{p{=}\mathrm{T},\ q{=}\mathrm{F}\} \dashv p \leftrightarrow q)$ is F because p and q do not have the same value. But $(\{p{=}\mathrm{F},\ q{=}\mathrm{F}\} \dashv p \leftrightarrow q)$ is T because p and q have the same value.

8. $(a \dashv \neg A)$ is T if $(a \dashv A)$ is F, and is F is $(a \dashv A)$ is T.

9. $(a \dashv (A)) = (a \dashv A)$. Parentheses only influence the structure of a propositional formula. A parenthesized formula (A) has the same meaning as A.

You determine the value of a propositional formula by building up larger and larger subexpressions, being careful to follow the rules of precedence and associativity. For example, suppose that $a = \{p{=}\mathrm{F},\ q{=}\mathrm{T},\ r{=}\mathrm{T}\}$. Then

(a) $(a \dashv q) = \mathrm{T}$

(b) $(a \dashv p) = \mathrm{F}$

(c) $(a \dashv \neg p) = \mathrm{T}$ by (b)

(d) $(a \dashv \neg p \wedge q) = \mathrm{T}$ by (a) and (c)

1.3 Implication

Intuitively, $A \to B$ means "if A is true then B is true." But that is not its definition. Its definition is that either A is false or B is true (or both). Notice that, if B is true, then $A \to B$ is true, *by definition*. Also, if A is false, then $A \to B$ is true, *by definition*.

1.4 Truth Tables

Since the value of a propositional formula depends on the values of its variables, one way to understand what the formula means is to look at its value for all possible values of the variables. That leads to the idea of a *truth table* of a propositional formula. The following is a truth table for $\neg p \vee q$.

p	q	$\neg$	p	$\vee$	q
F	F	T	F	T	F
F	T	T	F	T	T
T	F	F	T	F	F
T	T	F	T	T	T

Under each variable, we write that variable's value. Under each operator, we write the value of the formula having that operator as its main or outermost operator. The column in blue is the value of the entire formula, $\neg p \vee q$.

1.5 Validity

Definition 1.8. Propositional formula ϕ is *valid* if $(a \dashv \phi)$ is true for every truth value assignment a. A valid formula is also called a *tautology*.

For example, operator $\vee$ is commutative. Another way to say that is to say that formula

$$(P \vee Q) \leftrightarrow (Q \vee P)$$

is valid. Let's check that using a truth table.

p	q	$(p$	$\vee$	$q)$	$\leftrightarrow$	$(q$	$\vee$	$p)$
F	F	F	F	F	T	F	F	F
F	T	F	T	T	T	T	T	F
T	F	T	F	F	T	F	T	T
T	T	T	T	T	T	T	T	T

The validity of

$$(p \vee q) \ \leftrightarrow \ (q \vee p)$$

is evident from the blue column of all T's.

Table 1.9 shows a collection of propositional formulas that are all valid. It is worth noting that $\neg(p \rightarrow q)$ is equivalent to $p \wedge \neg q$. That is, $p \rightarrow q$ is false exactly when p is true and q is false. We will need that when doing proofs by contradiction.

Valid equivalences give you a way to replace one formula by another. For example, if you see $p \vee q$ in any context, you can replace it by $q \vee p$.

In fact, you can replace any variable by any propositional formula in any of the above tautologies (or any other valid propositional formula) and they are still valid, provided (1) you replace every occurrence of a variable by the same propositional formula and (2) you use parentheses to avoid rules of precedence from rearranging the formula. For example, the commutative law for $\wedge$ says that

$$p \wedge q \leftrightarrow q \wedge p.$$

Replacing p by $(w \rightarrow v)$ and q by $\neg r$ yields

$$(w \rightarrow v) \wedge \neg R \leftrightarrow \neg R \wedge (w \rightarrow v)$$

which is also valid.

Thinking ahead. You can determine whether a propositional formula is valid using a truth table. If the formula has n variables, the truth table has 2^n lines. That is not a problem when n is small, but what if your formula has 100 variables? 2^{100} is gigantic! An obvious question is: Does there exist an algorithm to determine whether a propositional formula is valid that is efficient enough to be used on long formulas that have a lot of variables? We will come back to this problem at the end of this term.

Table 1.9: Some propositional tautologies	
Equivalence	**Name**
$\neg(\neg p) \leftrightarrow p$	double negation
$p \vee q \leftrightarrow (q \vee p)$	commutative law of $\vee$
$p \wedge q \leftrightarrow (q \vee p)$	commutative law of $\wedge$
$(p \vee q) \vee r \leftrightarrow p \vee (q \vee r)$	associative law of $\vee$
$(p \wedge q) \wedge r \leftrightarrow p \wedge (q \wedge r)$	associative law of $\wedge$
$(p \wedge (q \vee r) \leftrightarrow (p \vee q) \wedge (p \vee r)$	distributive law of $\wedge$ over $\vee$
$(p \vee (q \wedge r) \leftrightarrow (p \wedge q) \vee (p \wedge r)$	distributive law of $\vee$ over $\wedge$
$\neg(p \vee q) \leftrightarrow \neg p \wedge \neg q$	DeMorgan's law for $\vee$
$\neg(p \wedge q) \leftrightarrow \neg p \vee \neg q$	DeMorgan's law for $\wedge$
$\neg(p \rightarrow q) \leftrightarrow p \wedge \neg q$	DeMorgan's law for $\rightarrow$
$p \rightarrow q \leftrightarrow \neg q \rightarrow \neg p$	Law of the contrapositive
$(p \vee q) \rightarrow r \leftrightarrow (p \rightarrow r) \wedge (q \rightarrow r)$	cases
$(p \wedge q) \rightarrow r \leftrightarrow (p \rightarrow (q \rightarrow r))$	
$p \wedge \neg p \leftrightarrow \mathbf{F}$	contradiction 1
$p \leftrightarrow (\neg p \rightarrow p)$	contradiction 2
$p \leftrightarrow (\neg p \rightarrow \mathbf{F})$	contradiction 3
$p \vee \neg p$	Law of the excluded middle
$p \rightarrow p$	Law of the excluded middle, restated using $\rightarrow$
$\neg(p \wedge \neg p)$	Law of the excluded middle (DeMogan variant)
$p \rightarrow (q \rightarrow p)$	
$\neg p \rightarrow (p \rightarrow q)$	

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