CSCI 4602
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Lecture Notes

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1 Review of Propositional Logic

This section reviews propositional logic, which you should already have seen.

BIG IDEA: Logic lies at the heart of mathematical reasoning. It is also essential for an understanding of how computers work.

1.1 Syntax of Propositional Logic

A *propositional formula* is an expression of propositional logic, such as $p \to q$. A propositional formula is also called a compound proposition. I will use the term propositional formula.

The *syntax* of propositional logic only says what a propositional formula looks like. It does not say what a propositional formula means. We use A, B, C and ϕ (Greek letter phi) to name arbitrary propositional formulas.

Definition 1.1. A *propositional formula* is defined as follows.

1. Symbols **T** and **F** are propositional formulas.
2. A *propositional variable* is a propositional formula. We will use p, q, r and s, possibly with subscripts, as propositional variables and X to refer to an arbitrary variable.
3. If A and B are propositional formulas then so are
   (a) $A \vee B$,
   (b) $A \wedge B$,
   (c) $A \to B$,
   (d) $A \leftrightarrow B$,
   (e) $\neg A$,
   (f) (A).

For example, each of the following is a propositional formula.

- p
- $p \vee q$
- $p \wedge \neg q$
- $p \wedge q \wedge r$
- $q \vee p \wedge r$
- $(r \wedge \mathbf{T}) \vee \neg q$
- $(p \rightarrow (q \rightarrow p))$

Operator $\vee$ is read "or", $\wedge$ is read "and", $\rightarrow$ is read "implies", $\leftrightarrow$ is read "if and only if" and $\neg$ is read "not".

1.1.1 Precedence and Associativity

Rules of *precedence* and *associativity* determine how you break a propositional formula into subformulas. Higher precedence operators are done first. The following lists operators by precedence, from highest to lowest.

Precedence	
parentheses	high
$\neg$	
$\wedge$	
$\vee$	
$\rightarrow$	
$\leftrightarrow$	low

For example, $p \vee q \wedge r$ is understood to have the same structure as $p \vee (q \wedge r)$ since $\wedge$ has higher precedence than $\vee$.

Associativity determines how an expression is broken into subexpressions when it involves two or more occurrences of the same operator. We assume

that operators $\vee$ and $\wedge$ are done from left to right. That is, they are *left-associative*. (Associativity is like the wind. A north wind blows from north to south.) For example, $p \vee q \vee r$ has the same structure as $(p \vee q) \vee r$. Associativity does not really matter for $\vee$ and $\wedge$ because they are *associative operators*. That is, $(p \vee q) \vee r$ and $p \vee (q \vee r)$ always have the same value.

Associativity does matter for some operators, so it is wise to think about it. By convention, operators $\rightarrow$ and $\leftrightarrow$ are done from right to left. So $p \rightarrow q \rightarrow r$ has the same meaning as $p \rightarrow (q \rightarrow r)$. However, we will always parenthesize when the associativity of $\rightarrow$ and $\leftrightarrow$ would be needed if the parentheses were omitted, to avoid confusion.

1.2 Meaning of Propositional Logic

The meaning of a propositional formula can only be defined when the values of all of its variables are given. Each variable can be true or false.

Definition 1.2. A *truth-value assignment* is a set of components of the form $X = V$ where X is a variable and V is either T or F. For example, $\{p$=T, q=F$\}$ is a truth-value assignment.

Definition 1.3. If a is a truth-value assignment and X is a variable then $a(X)$ is the value (T or F) that a gives for variable X. For example, if a is $\{p$=T, $q1$=F$\}$ then $a(p)$ = T and $a(q)$ = F.

Definition 1.4. Suppose that ϕ is a propositional formula and a is a truth-value assignment that defines every variable that occurs in ϕ. Notation $(a \dashv \phi)$ indicates the value of ϕ (either T or F) when variables have values given by a. Specifically:

1. $(a \dashv \mathbf{T})$ = T. That is, symbol **T** is always true.
2. $(a \dashv \mathbf{F})$ = F. That is, symbol **F** is always false.
3. If X is a variable then $(a \dashv X) = a(X)$. That is, X has the value that it is given by truth-value assignment a.
4. $(a \dashv A \vee B)$ is T if at least one of $(a \dashv A)$ and $(a \dashv B)$ is T, and is F otherwise. For example, $(\{p$=T, q=F$\} \dashv p \vee q)$ is T because $(\{p$=T, q=F$\} \dashv p)$ is T, and we only need one of p and q to be true.

5. $(a \dashv A \wedge B)$ is T if both of $(a \dashv A)$ and $(a \dashv B)$ are T, and is F otherwise. For example, $(\{p{=}\mathrm{T}, q{=}\mathrm{F}\} \dashv p \wedge q)$ is F because $(\{p{=}\mathrm{T}, q{=}\mathrm{F}\} \dashv p)$ and $(\{p{=}\mathrm{T}, q{=}\mathrm{F}\} \dashv q)$ are not both T.

6. $(a \dashv A \rightarrow B)$ has the same meaning as $(\neg A) \vee B$. Implication is discussed further below.

7. $(a \dashv A \leftrightarrow B)$ is true if $(a \dashv A)$ and $a \dashv B$ have the same value. For example, $(\{p{=}\mathrm{T}, q{=}\mathrm{F}\} \dashv p \leftrightarrow q)$ is F because p and q do not have the same value. But $(\{p{=}\mathrm{F}, q{=}\mathrm{F}\} \dashv p \leftrightarrow q)$ is T because p and q have the same value.

8. $(a \dashv \neg A)$ is T if $(a \dashv A)$ is F, and is F is $(a \dashv A)$ is T.

9. $(a \dashv (A)) = (a \dashv A)$. Parentheses only influence the structure of a propositional formula. A parenthesized formula (A) has the same meaning as A.

You determine the value of a propositional formula by building up larger and larger subexpressions, being careful to follow the rules of precedence and associativity. For example, suppose that $a = \{p{=}\mathrm{F}, q{=}\mathrm{T}, r{=}\mathrm{T}\}$. Then

(a) $(a \dashv q) = \mathrm{T}$

(b) $(a \dashv p) = \mathrm{F}$

(c) $(a \dashv \neg p) = \mathrm{T}$ by (b)

(d) $(a \dashv \neg p \wedge q) = \mathrm{T}$ by (a) and (c)

1.3 Implication

Intuitively, $A \rightarrow B$ means "if A is true then B is true." But that is not its definition. Its definition is that either A is false or B is true (or both). Notice that, if B is true, then $A \rightarrow B$ is true, *by definition.* Also, if A is false, then $A \rightarrow B$ is true, *by definition.*

1.4 Truth Tables

Since the value of a propositional formula depends on the values of its variables, one way to understand what the formula means is to look at its value for all possible values of the variables. That leads to the idea of a *truth table* of a propositional formula. The following is a truth table for $\neg p \vee q$.

p	q	$\neg$	p	$\vee$	q
F	F	T	F	T	F
F	T	T	F	T	T
T	F	F	T	F	F
T	T	F	T	T	T

Under each variable, we write that variable's value. Under each operator, we write the value of the formula having that operator as its main or outermost operator. The column in blue is the value of the entire formula, $\neg p \vee q$.

1.5 Validity

Definition 1.8. Propositional formula ϕ is *valid* if $(a \dashv \phi)$ is true for every truth value assignment a. A valid formula is also called a *tautology*.

For example, operator $\vee$ is commutative. Another way to say that is to say that formula

$$(P \vee Q) \ \leftrightarrow \ (Q \vee P)$$

is valid. Let's check that using a truth table.

p	q	$(p$	$\vee$	$q)$	$\leftrightarrow$	$(q$	$\vee$	$p)$
F	F	F	F	F	T	F	F	F
F	T	F	T	T	T	T	T	F
T	F	T	F	F	T	F	T	T
T	T	T	T	T	T	T	T	T

The validity of

$$(p \vee q) \;\leftrightarrow\; (q \vee p)$$

is evident from the blue column of all T's.

Table 1.9 shows a collection of propositional formulas that are all valid. It is worth noting that $\neg(p \rightarrow q)$ is equivalent to $p \wedge \neg q$. That is, $p \rightarrow q$ is false exactly when p is true and q is false. We will need that when doing proofs by contradiction.

Valid equivalences give you a way to replace one formula by another. For example, if you see $p \vee q$ in any context, you can replace it by $q \vee p$.

In fact, you can replace any variable by any propositional formula in any of the above tautologies (or any other valid propositional formula) and they are still valid, provided (1) you replace every occurrence of a variable by the same propositional formula and (2) you use parentheses to avoid rules of precedence from rearranging the formula. For example, the commutative law for $\wedge$ says that

$$p \wedge q \leftrightarrow q \wedge p.$$

Replacing p by $(w \rightarrow v)$ and q by $\neg r$ yields

$$(w \rightarrow v) \wedge \neg R \leftrightarrow \neg R \wedge (w \rightarrow v)$$

which is also valid.

Thinking ahead. You can determine whether a propositional formula is valid using a truth table. If the formula has n variables, the truth table has 2^n lines. That is not a problem when n is small, but what if your formula has 100 variables? 2^{100} is gigantic! An obvious question is: Does there exist an algorithm to determine whether a propositional formula is valid that is efficient enough to be used on long formulas that have a lot of variables? We will come back to this problem at the end of this term.

Table 1.9: Some propositional tautologies	
Equivalence	**Name**
$\neg(\neg p) \leftrightarrow p$	double negation
$p \vee q \leftrightarrow (q \vee p)$	commutative law of $\vee$
$p \wedge q \leftrightarrow (q \vee p)$	commutative law of $\wedge$
$(p \vee q) \vee r \leftrightarrow p \vee (q \vee r)$	associative law of $\vee$
$(p \wedge q) \wedge r \leftrightarrow p \wedge (q \wedge r)$	associative law of $\wedge$
$(p \wedge (q \vee r) \leftrightarrow (p \vee q) \wedge (p \vee r)$	distributive law of $\wedge$ over $\vee$
$(p \vee (q \wedge r) \leftrightarrow (p \wedge q) \vee (p \wedge r)$	distributive law of $\vee$ over $\wedge$
$\neg(p \vee q) \leftrightarrow \neg p \wedge \neg q$	DeMorgan's law for $\vee$
$\neg(p \wedge q) \leftrightarrow \neg p \vee \neg q$	DeMorgan's law for $\wedge$
$\neg(p \rightarrow q) \leftrightarrow p \wedge \neg q$	DeMorgan's law for $\rightarrow$
$p \rightarrow q \leftrightarrow \neg q \rightarrow \neg p$	Law of the contrapositive
$(p \vee q) \rightarrow r \leftrightarrow (p \rightarrow r) \wedge (q \rightarrow r)$	cases
$(p \wedge q) \rightarrow r \leftrightarrow (p \rightarrow (q \rightarrow r))$	
$p \wedge \neg p \leftrightarrow \mathbf{F}$	contradiction 1
$p \leftrightarrow (\neg p \rightarrow p)$	contradiction 2
$p \leftrightarrow (\neg p \rightarrow \mathbf{F})$	contradiction 3
$p \vee \neg p$	Law of the excluded middle
$p \rightarrow p$	Law of the excluded middle, restated using $\rightarrow$
$\neg(p \wedge \neg p)$	Law of the excluded middle (DeMogan variant)
$p \rightarrow (q \rightarrow p)$	
$\neg p \rightarrow (p \rightarrow q)$	

2 Review of First-Order Logic

First-order logic (also called *predicate logic*) is an extension of propositional logic that is much more useful than propositional logic. It was created as a way of formalizing common mathematical reasoning. You should have seen first-order logic previously. This section intended to be review.

In first-order logic, you start with a nonempty set of values called the *universe of discourse* U. Logical statements talk about properties of values in U and relationships among those values.

2.1 Predicates

In place of propositional variables, first-order logic uses *predicates*.

Definition 2.1. A *predicate* P takes zero or more parameters $x_1, x_2, \ldots, x_n$ and yields either true or false. First-order formula $P(x_1, \ldots, x_n)$ is the value of predicate P with parameters $x_1, \ldots, x_n$. A predicate with no parameters is a propositional variable equivalent to a propositional variable.

Suppose that U is the set of all integers. Here are some examples of predicates. There is no standard collection of predicates that are always used. Rather, each of these is like a function definition in a computer program; different programs contain different functions.

- We might define $\text{even}(n)$ to be true if n is even. For example $\text{even}(4)$ is true and $\text{even}(5)$ is false.
- We might define $\text{greater}(x, y)$ to be true if $x > y$. For example, $\text{greater}(7, 3)$ is true and $\text{greater}(3, 7)$ is false.
- We might define $\text{increasing}(x, y, z)$ to be true if $x < y < z$. For example, $\text{increasing}(2, 4, 6)$ is true and $\text{increasing}(2, 4, 2)$ is false.

We allow binary relations such $=$ and $<$ as predicates. For example, $x = y \rightarrow y < z$ is a formula of first-order logi.

2.2 Terms

A *term* is an expression that stands for a particular value in U. The simplest kind of term is a *variable*, which can stand for any value in U.

A *function* takes zero or more parameters that are members of U and yields a member of U. The following are examples of functions that might be defined when U is the set of all integers.

- A function with no parameters is called a *constant*. We allow constants such as 0 and 1.
- We might define $\text{successor}(n)$ to be $n+1$. For example, $\text{successor}(2) = 3$.
- We might define $\text{sum}(m, n)$ to be $m+n$. For example, $\text{sum}(5, 7) = 12$.
- We might define $\text{largest}(a, b, c)$ to be the largest of a, b and c. For example, $\text{largest}(3, 9, 4) = 9$ and $\text{largest}(4, 4, 4) = 4$.

Definition 2.2. A *term* is defined as follows.

1. A *variable* is a term. We use single letters such as x and y for variables.
2. If f is a function that takes no parameters then f is a term (standing for a value in U).
3. If f is a function that takes $n > 0$ parameters and $t_1, \ldots, t_n$ are terms then $f(t_1, \ldots, t_n)$ is a term.

For example, $\text{sum}(\text{sum}(x, y), \text{successor}(z))$ is a term.

We allow notation such as $x + y$ and $x - y$ as terms. That is, a function can be written as a binary operator. That is just a notational convenience.

The meaning of a term should be clear, provided the values of variables are known. Term $\text{sum}(x, y)$ stands for the result that function sum yields on parameters (x, y) (the sum of x and y).

2.3 First-Order Formulas

Definition 2.3. A *first-order formula* is defined as follows.

1. **T** and **F** are first-order formulas.
2. If P is a predicate that takes no parameters then P is a first-order formula.
3. If $t_1, \ldots, t_n$ are terms and P is a predicate that takes $n > 0$ parameters, then $P(t_1, \ldots, t_n)$ is a first-order formula. It is true if $P(v_1, \ldots, v_n)$ is true, where v_1 is the value of term t_1, v_2 is the value of term t_2, etc.
4. If t_1 and t_2 are terms then $t_1 = t_2$ is a first-order formula. (It is true if terms t_1 and t_2 have the same value.)
5. If A and B are first-order formulas and x is a variable then each of the following is a first-order formula.
   (a) (A)
   (b) $\neg A$
   (c) $A \vee B$
   (d) $A \wedge B$
   (e) $A \rightarrow B$
   (f) $A \leftrightarrow B$
   (g) $\forall x\, A$
   (h) $\exists x\, A$

The meaning of parentheses, **T**, **F**, $\neg$, $\vee$, $\wedge$, $\rightarrow$ and $\leftrightarrow$ are the same as in propositional logic. Symbols $\forall$ and $\exists$ are called *quantifiers*. You read $\forall x$ as "for all x", and $\exists x$ as "for some x" or "there exists an x". They have the following meanings.

1. $\forall x\, A$ is true of A is true for all values of x in U.
2. $\exists x\, A$ is true if A is true for at least one value of x in U.

By convention, quantifiers have higher precedence than all of the operators $\wedge$, $\vee$, etc.

Examples of first-order formulas are:

1. $P(\text{sum}(x, y))$ says that, if $v = \text{sum}(x, y)$, then $P(v)$ is true. Its value (true or false) depends on the meanings of predicate P and function sum, as well as on the values of variables x and y.

2. $\forall x(\text{greater}(x, x))$ says that $\text{greater}(x, x)$ is true for every value x in U. Using the meaning of $\text{greater}(a, b)$ given above, $\forall x(\text{greater}(x, x))$ is clearly false, since no x can be greater than itself.

3. $\neg\forall x(\text{greater}(x, x))$ says that $\forall x(\text{greater}(x, x))$ is false. That is true.

4. $\exists y(y = \text{sum}(y, y))$ says that there exists a value y where $y = y + y$. That is true since $0 = 0 + 0$.

5. $\forall x(\exists y(\text{greater}(y, x)))$ says that, for every value v of x, first-order formula $\exists y(\text{greater}(y, v))$ is true. That is true. If $v = 100$, then choose $y = 101$, which is larger than 100. If $v = 1000$, choose $y = 1001$. If $v = 1,000,000$, choose $y = 1,000,001$.

6. $\exists y(\forall x(\text{greater}(y, x)))$ says that there exists a value v of y so that $\forall x(\text{greater}(v, x))$. That is false. There is no single value v that is larger than every integer x.

3 Theorems and Proofs

A *theorem* is any mathematical statement, such as a formula of first-order logic, that has been proved true or is about to be proved true. (When you are about to prove a theorem, you call it theorem as a way of promising that a proof is about to be produced.)

BIG IDEA: Doing proofs teaches you to reason carefully and to present your ideas precisely. Writing computer software requires you to reason carefully and to present your ideas (in the form of a computer program) precisely. People who can do proofs are prepared for the rigors of software development.

3.1 What Is a Proof?

There are many different precise definitions of a proof. But most mathematicians accept an informal definition: a proof is a clear and unambiguous argument that a mathematical statement is true and that any sufficiently knowledgeable person can check. The key is that a reader must be able to check that each step in the proof is correct.

Students who are just learning to do proofs make many different kinds of mistakes, but most fall into one of the following two categories.

1. **The student does not check his or her own work.** The reason for this can vary from lack of time to lack of understanding to fear of failure.

   Errors in elementary algebraic manipulations are surprisingly common. Simply looking at your work with a critical eye will usually suffice to find those errors.

   When you do a proof, it is a good idea to look at proofs of similar statements that you have already seen, and to modify those proofs to work for your current goal. But be cautious! It is easy to end up with something that makes no sense. Check the modified proof carefully with a critical eye.

   The student who has a lack of understanding cannot check the proof. The only remedy is to gain the necessary understanding.

The student who is afraid of failure will not check the proof out of fear that it might turn out to be incorrect. That can be cured by adopting a policy of checking everything that you write and fixing it when you encounter an error. Take pride in your work and care whether it is right.

Regardless of your reason for not checking your proof, you can be sure that an unchecked proof is incorrect, for the same reasons that an untested computer program does not work. You will need to find a way to motivate yourself to check your proofs carefully.

2. Mathematics relies on precise definitions. When you do a proof, it is essential for you to use definitions wherever appropriate. **Students often get stuck in a proof because they have forgotten to use definitions**. Any time you cannot see how to proceed, ask yourself if using a definition will help. We will see examples of that.

3.2 Deus ex Machina

Deus ex machina is latin for "god from the machine." In ancient plays, there were certain rules that the playwright was required to follow. (Modern movies have similar constraints. You are not allowed to kill off the hero, for example, especially when the hero is played by a popular actor.) Sometimes the playwright got his characters in a real bind, and he could not see how to get them out of it. The solution was to have a platform rise up with an actor on it dressed as a god. The actor would wave his arms and fix everything.

I have frequently found that students resort to deus ex machina in their proofs. Having written part of a proof, the student finds that he or she cannot make further progress. So he or she simply writes the goal and claims that it has been proved. Voila!

I can only guess that a student does that in the hope of getting partial credit for the part of the "proof" before pulling the goal out of the air. But pulling the goal out of the air will *decrease* your score. It is better to admit that you are stuck. Even better, back up and try a different approach, or check your algebra; with correct algebra, you might not actually be in a bind at all.

3.3 Forward Proofs

A *forward proof* reasons from what you know to what you can conclude. The proof accumulates knowledge and (named) values until it reaches a point where the goal is known. Each new conclusion can rely on prior knowledge or conclusions.

You have probably been taught a different approach in an algebra class. In a *backwards proof*, you write down what you want to show and then perform some manipulations on it, working backwards to a statement that you already know is true, such as $x = x$.

In this class, we will do forward proofs, with small excursions that typically convert a goal into an equivalent goal, followed by a proof of the equivalent goal. **I expect you to use forward proofs as well.** At least for this class, put aside the backwards proofs that you have learned in algebra.

In this section, I do proofs at two different levels of detail. The first proof of a theorem works in small steps and shows everything that you know after each step. The second proof of the same theorem is more typical of what you would write, and what I want to see from you.

3.4 Some Definitions

Definition 3.1. Integer n is *even* if there exists an integer m such that $n = 2m$. For example, 6 is even because $6 = (2)(3)$.

Definition 3.2. Integer n is *odd* if there exists an integer m such that $n = 2m + 1$. We will also make use of the fact that, for every n, n is odd if and only if n is not even.

Definition 3.3. Integer n is a *perfect square* if there exists an integer m such that $n = m^2$.

Definition 3.4. Real number x is *rational* if there exist integers n and m where $m \neq 0$ such that $x = n/m$.

3.5 Proving A Implies B

You typically prove an implication by *direct proof*: **To prove $A \to B$, assume that A is true and show that B is true.** That is, add A to your knowledge. Then prove B.

Example Theorem 3.5. If n is even then n^2 is even.

Detailed Proof.

1. Suppose that n is even.

Known variables:	n
Know:	n is even.
Goal:	n^2 is even.

2. By the definition of an even integer, there exists an integer m such that $n = 2m$.

Known variables:	n, m
Know:	n is even.
Know:	$n = 2m$.
Goal:	n^2 is even.

3. Since $n = 2m$, $n^2 = (2m)^2 = 4m^2 = 2(2m^2)$.

Known variables:	n, m
Know:	n is even.
Know:	$n = 2m$.
Know:	$n^2 = 2(2m^2)$.
Goal:	n^2 is even.

4. So $n^2 = 2(x)$ where $x = 2m^2$. Using the definition of an even number again, n^2 is even.

$\diamondsuit$

Typical Proof. Suppose n is even. By the definition of an even integer, there is an integer m such that $n = 2m$. So

$$n^2 = (2m)^2 = 4m^2 = 2(2m^2).$$

By the definition of an even integer, n^2 is even.

$\diamondsuit$

Example Theorem 3.6. If n and m are perfect squares then nm is a perfect square.

Detailed Proof.

1. Suppose that n and m are perfect squares.

Known variables:	n, m
Know:	n is a perfect square.
Know:	m is a perfect square.
Goal:	nm is a perfect square.

2. By the definition of a perfect square, there exist integers x and y such that $n = x^2$ and $m = y^2$.

Known variables:	n, m, x, y
Know:	$n = x^2$.
Know:	$m = y^2$.
Goal:	nm is a perfect square.

3. Replacing n by x^2 and m by y^2, $nm = x^2y^2 = (xy)^2$.

Known variables:	n, m, x, y
Know:	$n = x^2$.
Know:	$m = y^2$.
Know:	$nm = (xy)^2$.
Goal:	nm is a perfect square.

4. So $nm = z^2$ where $z = xy$. Using the definition of a perfect square again, nm is perfect square.

♢

Typical Proof. Suppose that n and m are perfect squares. By the definition of a perfect square, there exist integers x and y such that $n = x^2$ and $m = y^2$. Replacing n by x^2 and m by y^2,

$$nm = x^2y^2 = (xy)^2.$$

So nm is a perfect square.

♢

3.5.1 Using the Contrapositive

You can prove any theorem by proving an equivalent mathematical statement. For example, you can prove $A \to B$ by proving equivalent formula $\neg B \to \neg A$, which is called the *contrapositive* of $A \to B$. Here is an example.

Example Theorem 3.7. Suppose n is an integer. If $3n + 2$ is odd, then n is odd.

Detailed Proof. We prove the contrapositive: If n is not odd then $3n + 2$ is not odd.

1. We know that an integer x is even if and only if x is not odd. So what we want to prove is equivalent to: If n is even then $3n + 2$ is even.

Known variables:	n
Goal:	If n is even then $3n + 2$ is even.

2. Suppose that n is even.

Known variables:	n
Know:	n is even.
Goal:	$3n + 2$ is even.

3. By the definition of an even integer, there exists an integer m such that $n = 2m$.

Known variables:	n, m
Know:	$n = 2m$.
Goal:	$3n + 2$ is even.

4. $3n + 2 = 3(2m) + 2 = 6m + 2 = 2(3m + 1)$.

Known variables:	n, m
Know:	$n = 2m$.
Know:	$3n + 2 = 2(3m + 1)$.
Goal:	$3n + 2$ is even.

5. Using the definition of an even integer again, $3n + 2$ is even because $3n + 2 = 2z$ where $z = 3m + 1$.

$\diamondsuit$

Typical Proof. We prove the contrapositive: If n even then $3n+2$ is even.

Suppose n is even. Then there exists an integer m such that $n = 2m$.

$$3n + 2 = 3(2m) + 2 = 6m + 2 = 2(3m + 1).$$

Since $3n + 2$ is twice an integer, $3n + 2$ is even.

$\diamondsuit$

3.6 Proving and Using (A and B)

To prove $A \wedge B$, prove A and prove B.

If you know that $A \wedge B$ is true, then you know that A is true and you know that B is true.

3.7 Proving and Using not(A)

To prove $\neg(A)$, you typically use DeMorgan's laws and the laws for negating quantified formulas to push the negation inward. For example, to prove $\neg(A \wedge B)$, you prove equivalent formula $\neg A \vee \neg B$. To prove $\neg(\forall x A)$, you prove equivalent formula $\exists x(\neg A)$.

The same principle applies when you already know $\neg(A)$. For example, if you know $\neg(A \to B)$, you can conclude equivalent formula $A \wedge \neg B$. You write that down as an additional known fact.

3.8 Proving and Using (A or B)

To prove $A \vee B$, you usually prove one of the equivalent formulas $\neg A \to B$ or $\neg B \to A$.

Suppose that you know that $A \vee B$ is true and you want to use that to show that C is true. That is, you want to show that $A \vee B \to C$ is true. You typically prove equivalent formula

$$A \to C \;\wedge\; B \to C.$$

That is called *proof by cases*. First, you assume that A is true and show that C is true. Next, you assume that B is true and show that C is true. See Section 3.13.

Proof by cases requires you to do each case in a silo. Assumptions made while proving $A \to C$ cannot be used when proving $B \to C$. When you start to prove $B \to C$, your knowledge reverts to what it was when you were just about to start to prove $A \to C$.

3.9 Proving and Using Existential Statements

To prove that something exists, produce it. That is called a *constructive existence proof.*

Example Theorem 3.8. There exists an integer n where n is even and n is prime.

Proof. Choose $n = 2$. Notice that n is even and n is prime.

$\diamondsuit$

3.9.1 Using Existential Knowledge

Sometimes, instead of needing to prove $\exists x P(x)$, you already know $\exists x P(x)$. What do you do? You ask somebody else to give you a value x so that $P(x)$ is true. It is not necessary for you to say how to find x. We will encounter many examples of that.

3.10 Proving Universal Statements

To prove $\forall x P(x)$, prove $P(x)$ for an *arbitrary* value of x.

That does not mean that you can choose the value of x. Rather, someone else chooses x and you must prove that $P(x)$ is true for that value of x. Think of it as a challenge. You say to someone else, give me any value of x that you like. I will prove that $P(x)$ is true. In mathematics, *arbitrary* always means a value chosen by someone else.

We have actually used this idea above. When the statement of a theorem involves unbound (unquantified) variables, it is assumed to be saying that the statement is true for all values of those variables. Here is the first proof above with the quantifier explicit. The universe of discourse is the set of all integers.

Example Theorem 3.9. $\forall n(n \text{ is even} \to n^2 \text{ is even})$.

Detailed Proof.

1. Ask someone else to select an arbitrary integer n. (We cannot assume anything about n except that it belongs to the universe of discourse.) We must prove: (n is even $\to n^2$ is even) for that n.

Known variables:	n
Goal:	n is even $\to n^2$ is even.

2. Suppose that n is even.

Known variables:	n
Know:	n is even.
Goal:	n^2 is even.

3. By the definition of an even integer, there exists an integer m such that $n = 2m$.

Known variables:	n
Know:	$\exists m(n = 2m)$.
Goal:	n^2 is even.

4. Ask someone else to provide the integer m that is asserted to exist.

Known variables:	n, m
Know:	$n = 2m$.
Goal:	n^2 is even.

5. Since $n = 2m$, $n^2 = (2m)^2 = 4m^2 = 2(2m^2)$.

Known variables:	n, m
Know:	$n = 2m$.
Know:	$n^2 = 2(2m^2)$.
Goal:	n^2 is even.

6. So $n = 2(x)$ where $x = 2m^2$. Using the definition of an even number again, n is even.

$\diamondsuit$

Typical Proof. Let n be an arbitrary even integer. By the definition of an even integer, there exists an integer m such that $n = 2m$. So

$$n^2 = (2m)^2 = 4m^2 = 2(2m^2).$$

Evidently, n^2 is even.

$\diamondsuit$

3.10.1 Proof by Contradiction

You can prove any theorem by proving an equivalent theorem. We have seen propositional tautology

$$p \leftrightarrow (\neg p \rightarrow \mathbf{F}).$$

That is, to prove p, assume that p is false and prove that $\mathbf{F}$ is true. Typically, you use the tautology

$$(q \wedge \neg q) \leftrightarrow \mathbf{F}$$

to prove $\mathbf{F}$ by proving a statement and its negation. That is called *proof by contradiction*. Let's use proof by contradition to reprove a theorem that we proved above.

Example Theorem 3.10. For every integer n, if $3n+2$ is odd, then n is odd.

Detailed Proof.

1. Reasoning by contradiction, we can assume the theorem is false and prove $\mathbf{F}$. That is:

Know:	$\neg\forall n(3n+2 \text{ is odd } \rightarrow n \text{ is odd})$.
Goal:	$\mathbf{F}$.

2. We can push the negation across the quantifier using valid formula $\neg\forall x A \leftrightarrow \exists x(\neg A))$.

Know:	$\exists n(\neg(3n+2 \text{ is odd } \rightarrow n \text{ is odd}))$.
Goal:	$\mathbf{F}$.

3. Now use the tautology that $\neg(p \rightarrow q) \leftrightarrow p \wedge \neg q)$.

Know:	$\exists n(3n+2 \text{ is odd } \wedge n \text{ is even})$.
Goal:	$\mathbf{F}$.

4. Ask somebody else to select an integer n such that $3n+2$ is odd and n is even.

Known variables:	n
Know:	$3n+2$ is odd.
Know:	n is even.
Goal:	$\mathbf{F}$.

5. By the definition of an even integer, saying that n is even is equivalent to saying that there exists an integer m such that $n = 2m$. (Existential information is useful because it allows you to get something in hand, as is done in the next step. So you often want to exploit existential information.)

Known variables:	n
Know:	$3n + 2$ is odd.
Know:	$\exists m(n = 2m)$.
Goal:	**F**.

6. Since we know that an integer m exists such that $n = 2m$, we can ask somebody else to give us such an m. Let's do that.

Known variables:	n, m
Know:	$3n + 2$ is odd.
Know:	$n = 2m$.
Goal:	**F**.

7. Since we know that $n = 2m$, it seems reasonable to substitute $2m$ for n in expression $3n + 2$ to see what we get. Doing that gives

$$3n + 2 = 3(2m) + 2 = 6m + 2 = 2(3m + 1).$$

So $3n + 2$ is even. Recording that:

Known variables:	n, m
Know:	$3n + 2$ is odd.
Know:	$n = 2m$.
Know:	$3n + 2$ is even.
Goal:	**F**.

8. But $3n + 2$ cannot be both even and odd. Formula ($3n + 2$ is odd $\wedge$ $3n + 2$ is even) is equivalent to **F**. So we have concluded that **F** is true and we are done.

$\diamondsuit$

Typical Proof. By contradiction. Assume there exists an n such that $3n+2$ is odd but n even. Since n is even, there exists an integer m so that $n = 2m$. So

$$3n + 2 = 3(2m) + 2 = 6m + 2 = 2(3m + 1).$$

That means $3n + 2$ is even, contradicting the assumption that $3n + 2$ is odd.

$\diamondsuit$

3.11 Proving $\forall x(\exists y(A))$

It is common to encounter theorems whose general form is $\forall x(\exists y P(x, y))$. The proof usually involves finding an algorithm. For any x, the algorithm must find a y so that $P(x, y)$ is true. Here is an example.

Example Theorem 3.11. For all real numbers x and y, if x and y are both rational numbers then $x + y$ is also a rational number.

Detailed Proof.

1. Ask someone else to select arbitrary real numbers of x and y.

Known variables:	x, y
Goal:	If x and y are rational then $x + y$ is rational.

2. Assume that x and y are rational.

Known variables:	x, y
Know:	x is rational.
Know:	y is rational.
Goal:	$x + y$ is rational.

3. Our knowledge involves the term *rational*. We need to know what that means. From the definition of a rational number, there must exist integers a and b where $b \neq 0$ and $x = a/b$; and there must exist integers c and d where $d \neq 0$ and $y = c/d$. (Notice that different names are chosen for different things.)

Known variables:	x, y, a, b, c, d
Know:	a, b, c and d are integers.
Know:	$b \neq 0$.
Know:	$d \neq 0$.
Know:	$x = a/b$.
Know:	$y = c/d$.
Goal:	$x + y$ is rational.

4. Since the goal is to show that $x + y$ is rational, let's replace x by a/b and replace y by c/d in expression $x + y$. Since b and d are nonzero,

$$x + y = a/b + c/d = ad/bd + bc/bd = (ad + bc)/bd.$$

Known variables:	x, y, a, b, c, d
Know:	a, b, c and d are integers.
Know:	$b \neq 0$.
Know:	$d \neq 0$.
Know:	$x = a/b$.
Know:	$y = c/d$.
Know:	$x + y = (ad + bc)/bd$.
Goal:	$x + y$ is rational.

5. But we have shown that $x + y$ is the ratio of integers $ad + bc$ and bd. Since neither b nor d is 0, bd cannot be 0. So $x + y$ is rational, by the definition of a rational number.

   The algorithm that has been employed here is for adding two fractions.

$\diamondsuit$

Typical Proof. Let x and y be arbitrary rational numbers. By the definition of a rational number, there exists integers a, b, c and d ($b \neq 0$ and $d \neq 0$) such that $x = a/b$ and $y = c/d$. Then

$$x + y = a/b + c/d = ad/bd + bc/bd = (ad + bc)/bd.$$

Since $x + y$ is the ratio of two integers, $x + y$ is rational. (You can observe that $bd \neq 0$ since the product of two nonzero numbers is nonzero.)

3.12 Proving (A If and Only If B)

There are two commonly used ways of proving $A \leftrightarrow B$.

3.12.1 Using Direct Equivalences

You can treat $\leftrightarrow$ in a way similar to the way you treat $=$ in algebraic equations, performing equivalence-preserving manipulations. Let's use that approach to prove the law of the contrapositive.

Example Theorem 3.12. $p \to q \leftrightarrow \neg q \to \neg p$.

Proof.

$$\begin{array}{rcll} \neg q \to \neg p & \leftrightarrow & \neg(\neg q) \vee \neg p & \text{(defn of } \to) \\ & \leftrightarrow & q \vee \neg p & \text{(double negation)} \\ & \leftrightarrow & \neg p \vee q & \text{(commutative law of } \vee) \\ & \leftrightarrow & p \to q & \text{(defn of } \to) \end{array}$$

3.12.2 Proving Two Implications

Sometimes it is preferable to use the fact that $A \leftrightarrow B$, is equivalent to $(A \to B) \wedge (B \to A)$ and to prove $A \to B$ and $B \to A$ separately.

Example Theorem 3.13. For every integer n, n is odd if and only if n^2 is odd.

Detailed Proof.

1. It suffices to prove

$$\forall n((n \text{ is odd } \to n^2 \text{ is odd}) \ \wedge \ (n^2 \text{ is odd } \to n \text{ is odd})).$$

   That gives two goals. We use tautology $\forall x(A \wedge B) \leftrightarrow (\forall x A \wedge \forall x B)$ and change the variable names so that we can look at the two parts separately without variables from one interfering with the other.

Goal (1):	$\forall n(n \text{ is odd } \to n^2 \text{ is odd})$.
Goal (2):	$\forall m(m^2 \text{ is odd } \to m \text{ is odd})$.

2. Ask someone else to choose arbitrary values of m and n.

Known variables:	n, m
Goal (1):	n is odd $\rightarrow n^2$ is odd.
Goal (2):	m^2 is odd $\rightarrow m$ is odd.

3. Goal (2) is equivalent to its contrapositive, m is even $\rightarrow m^2$ is even. We proved that as Theorem 3.1. That only leaves Goal (1). (We still know goal (2), of course, but we can always discard known information to simplify.)

Known variables:	n
Goal (1):	n is odd $\rightarrow n^2$ is odd.

4. To prove Goal (1), assume that n is odd.

Known variables:	n
Know:	n is odd.
Goal (1):	n^2 is odd.

5. Since n is odd, there exists an integer k so that $n = 2k + 1$.

Known variables:	n
Know:	$\exists k(n = 2k + 1)$.
Goal (1):	n^2 is odd.

6. Ask someone else to provide a value k such that $n = 2k + 1$.

Known variables:	n, k
Know:	$n = 2k + 1$.
Goal (1):	n^2 is odd.

7. Since $n = 2k + 1$,

$$n^2 = (2k + 1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1.$$

Since $n^2 = 2z + 1$ for $z = 2k^2 + 2k$, it is evident that n^2 is odd.

♢

Typical Proof.

(a) (n is odd $\to n^2$ is odd) Assume that n is odd. By the definition of an odd integer, there is an integer k such that $n = 2k + 1$. So

$$n^2 = (2k + 1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1.$$

By the definition of an odd integer, n^2 is odd.

(b) (n^2 is odd $\to n$ is odd) This is equivalent to (n is even $\to n^2$ is even), which we profed earlier as Theorem 3.1.

♢

3.13 Proof by Cases

Proof by cases involves proving two or more statements. You must be careful that assumptions made during one of those cases are not still in place when proving another one. Think of this is similar to calling a function in a program. Each time a function is called, a new frame is created, so that calling $f(3)$ does not interfere with a later call to $f(4)$.

Example Theorem 3.14. For every integer n, $n^2 \geq n$.

Detailed Proof.

1. Ask someone to select an arbitrary integer n.

Known variables:	n
Know:	n is an integer
Goal:	$n^2 \geq n$.

2. Let's break proving the goal into three cases: $n = 0$, $n > 0$ and $n < 0$.

Known variables:	n
Know:	n is an integer
Goal (1):	$n = 0 \to n^2 \geq n.$
Goal (2):	$n > 0 \to n^2 \geq n.$
Goal (3):	$n < 0 \to n^2 \geq n.$
Goal (4):	$n^2 \geq n.$

3. Goal (1) is clearly true since $0^2 \geq 0$. Let's record it among the known facts.

Known variables:	n
Know:	n is an integer
Know (1):	$n = 0 \to n^2 \geq n.$
Goal (2):	$n > 0 \to n^2 \geq n.$
Goal (3):	$n < 0 \to n^2 \geq n.$
Goal (4):	$n^2 \geq n.$

4. Goal (2) is an implication, so we should assume that $n > 0$ and prove that $n^2 \geq n$. But let's prove that as a separate subproof. Knowledge and goals that are local to the proof of goal (2) are numbered 2.1, 2.2, etc., and they can only be used to establish goal (2).

Known variables:	n
Know:	n is an integer
Know (1):	$n = 0 \to n^2 \geq n.$
Goal (2):	$n > 0 \to n^2 \geq n.$
Goal (3):	$n < 0 \to n^2 \geq n.$
Goal (4):	$n^2 \geq n.$
Know (2.1):	$n > 0$
Goal (2.1):	$n^2 \geq n$

5. Since $n > 0$ is an integer, it must be the case that $n \geq 1$.

Known variables:	n
Know:	n is an integer
Know (1):	$n = 0 \to n^2 \geq n$.
Goal (2):	$n > 0 \to n^2 \geq n$.
Goal (3):	$n < 0 \to n^2 \geq n$.
Goal (4):	$n^2 \geq n$.
Know (2.1):	$n \geq 1$
Goal (2.1):	$n^2 \geq n$

Multiplying both sides of fact (2.1) by n preserves the inequality because $n > 0$. That gives $n \cdot n \geq n \cdot 1$, or equivalently, $n^2 \geq n$.

Known variables:	n
Know:	n is an integer
Know (1):	$n = 0 \to n^2 \geq n$.
Goal (2):	$n > 0 \to n^2 \geq n$.
Goal (3):	$n < 0 \to n^2 \geq n$.
Goal (4):	$n^2 \geq n$.
Know (2.1):	$n \geq 1$
Know (2.2):	$n^2 \geq n$
Goal (2.1):	$n^2 \geq n$

6. We have succeeded in proving goal (2). Notice that fact (2.2) cannot be used to establish goal (4) since it depends on the assumption that $n > 0$.

   We can move goal (2) into our knowledge. But we must also throw out parts that were local to the proof of goal (2).

Known variables:	n
Know:	n is an integer
Know (1):	$n = 0 \to n^2 \geq n$.
Know (2):	$n > 0 \to n^2 \geq n$.
Goal (3):	$n < 0 \to n^2 \geq n$.
Goal (4):	$n^2 \geq n$.

7. Now we need to prove goal (3). Assume that $n < 0$. But the square of any number is nonnegative. It follows that $n^2 \geq 0 > n$ when $n < 0$, and we can move goal (3) into what we know.

Known variables:	n
Know:	n is an integer
Know (1):	$n = 0 \rightarrow n^2 \geq n.$
Know (2):	$n > 0 \rightarrow n^2 \geq n.$
Know (3):	$n < 0 \rightarrow n^2 \geq n.$
Goal (4):	$n^2 \geq n.$

8. Propositional formula

$$(p \rightarrow s) \wedge (q \rightarrow s) \wedge (r \rightarrow s) \wedge (p \vee q \vee r)) \rightarrow s$$

is a tautology. That means known facts (1), (2) and (3) imply goal (4).

♢

Typical Proof. The proof is by cases ($n = 0$, $n > 0$ and $n < 0$).

Case 1 ($n = 0$). Then $n^2 \geq n$ because $0^2 \geq 0$.

Case 2 ($n > 0$). The smallest positive integer is 1, so $n > 0$ implies $n \geq 1$. Multiplying both sides of inequality $n \geq 1$ by positive number n gives $n^2 \geq n$.

Case 3 ($n < 0$). $n^2 \geq 0$ for all numbers n. Since, in this case, n is negative, clearly $n^2 \geq n$.

♢

4 Mathematical Foundations

4.1 Sets

You should have seen sets before. This is review.

Definition 4.1. A *set* is an unordered collection of things without repetitions. The things in set S are called the *members* of S.

Definition 4.2. A *set enumeration* is one way to describe a set, by writing the members of the set in braces, separated by commas. For example, {2, 5, 9} is a set of three integers.

4.1.1 Finite and Infinite Sets

It is possible to list the members of a *finite* set. But some sets, such as the set of all positive integers, have infinitely many members. Here are a few common infinite sets.

$\mathcal{N}$	$\{0, 1, 2, 3, \ldots\}$
$\mathcal{Z}$	$\{\ldots, -2, -1, 0, 1, 2, \ldots\}$
$\mathcal{R}$	the set of all real numbers

Note. Mathematicians usually define $\mathcal{N}$ to be the set of all *positive* integers. Computer scientists usually define $\mathcal{N}$ to be the set of nonnegative integers because 0 is so important to computer programs. I will use the definition above, including 0.

4.1.2 Set Comprehensions

A *set comprehension* is a way to describe the set of all values that have a certain property. Notation

$$\{x \mid p(x)\}$$

stands for the set of all values x such that $p(x)$ is true and notation

$$\{f(x) \mid p(x)\}$$

stands for the set of all values $f(x)$ such that $p(x)$ is true. Notation

$$\{x \in S \mid p(x)\}$$

is shorthand for $\{x \mid x \in S \wedge p(x)\}$ Here are some examples.

Set	**Description**
$\{x \mid x \in \mathcal{R} \ \wedge \ x^2 - 2x + 1 = 0\}$	$\{-1,\, 1\}$
$\{x \in \mathcal{R} \mid x^2 - 2x + 1 = 0\}$	$\{-1,\, 1\}$
$\{x \mid x$ is an even positive integer$\}$	$\{2, 4, 6, \ldots\}$
$\{x^2 \mid x$ is an even positive integer$\}$	$\{4, 16, 36, \ldots\}$

4.1.3 Set Notation and Operations

Table 4.1 defines notation for sets.

Note. Mathematicians commonly use operator $\setminus$ to mean set difference, and that is what Cummings does. That is, Cummings defines $S \setminus T$ to be the set of all members of S that are not members of T. Computer scientists usually write $S - T$ for set difference.

4.1.4 Identities for Sets

Table 4.2 lists some identities that are easy to establish.

4.1.5 Sets of Sets

The members of sets can be sets. For example, if $S = \{\{1, 2, 3\}, \{2, 4, 6\}\}$ then $|S| = 2$, since S has exactly two members, $\{1,\ 2,\ 3\}$ and $\{2,\ 4,\ 6\}$.

Do not confuse $\in$ with $\subseteq$. If $S = \{\{1, 2, 3\}, \{2, 4, 6\}\}$ then

$\{1, 2, 3\} \in S$

$\{1, 2, 3\} \not\subseteq S$

$3 \notin S$

Notice that $\{\} \neq \{\{\}\}$. $|\{\}| = 0$ but $|\{\{\}\}| = 1$ since $\{\{\}\}$ has one member, the empty set.

Table 4.1	
Notation	**Meaning**
$\lvert S \rvert$	The *cardinality* (size) of S, when S is a finite set.
$\{\}$	The empty set, which has no members
$x \in S$	True if x is a member of set S. For example, $2 \in \{1, 2, 3, 4\}$
$x \notin S$	$\neg(x \in S)$
$S \cup T$	$\{x \mid x \in S \vee x \in T\}$. For example, $\{2, 5, 6\} \cup \{2, 3, 7\} = \{2, 3, 5, 6, 7\}$. This is called the *union* of sets S and T.
$S \cap T$	$\{x \mid x \in S \wedge x \in T\}$. For example, $\{2, 5, 6\} \cup \{2, 3, 7\} = \{2\}$. This is called the *intersection* of sets S and T.
$S - T$	$\{x \mid x \in S \wedge x \notin T\}$. For example, $\{2, 5, 6\} - \{2, 3, 7\} = \{5, 6\}$. This is called the *difference* of sets S and T.
$\overline{S}$	$U - S$, where U is the universe of discourse. This is called the *complement* of S.
$S \times T$	$\{(x, y) \mid x \in S \wedge y \in T\}$. For example, $\{2, 3\} \times \{5, 6\} = \{(2,5), (2,6), (3,5), (3,6)\}$. This is called the *cartesian product* of S and T.
$S \subseteq T$	This is true if $\forall x(x \in S \rightarrow x \in T)$. For example, $\{2, 4, 6\} \subseteq \{1, 2, 3, 4, 5, 6\}$. Notice that $\{2, 4, 6\} \subseteq \{2, 4, 6\}$. $S \subseteq T$ is read "S is a subset of T."
$S = T$	S and T are the same set if $S \subseteq T$ and $T \subseteq S$. That is, S and T have exactly the same members.

Table 4.2
Some Set Identities
$A \cup \{\} = A$
$A \cap \{\} = \{\}$
$\overline{\overline{A}} = A$
$A \cup B = B \cup A$
$A \cap B = B \cap A$
$A \cup (B \cup C) = (A \cup B) \cup C$
$A \cap (B \cap C) = (A \cap B) \cap C$
$\overline{A \cup B} = \overline{A} \cap \overline{B}$
$\overline{A \cap B} = \overline{A} \cup \overline{B}$
$A - B = A \cap \overline{B}$.
$A \cup (A \cap B) = A$
$A \cap (A \cup B) = A$

4.2 Alphabets and Strings

Definition 4.3. An *alphabet* is a finite, nonempty set whose members we call *symbols*.

We will usually want to use small alphabets such as $\{a, b\}$ or $\{a, b, c\}$, where symbols a, b and c stand for themselves (letters of an alphabet).

It is conventional to call an alphabet Σ (upper case Greek letter sigma, indicating *symbol*).

Definition 4.4. If Σ is an alphabet, then a *string over* Σ is a finite sequence members of Σ. (In a sequence, order matters and there can be repetitions.) A string can have length 0.

I will write strings in double-quotes. For example, if $\Sigma = \{a, b, c\}$ then "aab" and "$ccccc$" are two strings over Σ.

A fundamental operations on strings is *concatenation*, where $s \cdot t$ indicates s followed by t. For example, "abc" $\cdot$ "aba" = "$abcaba$". Just as the multiplication symbol is usually unwritten between numbers, we will usually omit the concatenation dot between strings and write st to mean $s \cdot t$.

We will allow concatenation to work with symbols as well as strings. For example, "aab" $\cdot\, a$ = "$aaba$".

When the alphabet is understood or unimportant, we talk about a *string*, leaving the alphabet unstated.

Definition 4.5. If s is a string, then $|s|$ is the length of s (the number of characters in s). For example, $|$"$accb$"$| = 4$ and $|$"b"$| = 1$.

Definition 4.6. We write ε to mean the empty string, "", whose length is 0. (Symbol ε is a variant of Greek letter epsilon. Think of it as e for empty.)

4.2.1 Sets of Strings

Definition 4.7. A set of strings is called a *language*.

Definition 4.8. If Σ is an alphabet, then Σ^* is the set of all strings over Σ. For example, $\{a, b\}^* = \{\varepsilon,$ "a", "b", "aa", "ab", "ba", "bb", "aaa", $\ldots\}$.

4.2.2 Natural Numbers as Strings

We will use strings as the inputs and outputs of algorithms or programs. But sometimes, we want the inputs and outputs to be integers. That is easy to manage: we write the integers in standard (base 10) notation as strings. For example, 25 is treated as string is "25".

4.3 Functions

You should have seen functions before. This is review.

Definition 4.9. If A and B are sets, then a *function with domain A and codomain B* associates exactly one value in set B with each value in set A. We write $f : A \to B$ to mean that f is a function with domain A and codomain B.

Definition 4.10. If $f : A \to B$ and $x \in A$, then notation $f(x)$ indicates the member of B that f associates with x. When $f(x) = y$, we say that f *maps* x to y.

For example, suppose that $f : \mathcal{N} \to \mathcal{N}$ is defined by $f(x) = x^2$. Then $f(3) = 9$ and $f(5) = 25$.

4.4 Computational Problems

We will look at two kinds of computational problems.

1. A *decision problem* is a problem where the input is a string (over a chosen *input alphabet*) and the output is either 1 (true) or 0 (false). We can also think of the output as yes or no.

   A decision problem can be expressed as a function or as a set of strings (a language). When S is a set of strings, we think of S as the decision problem:

   > **Input.** String x over the input alphabet.
   >
   > **Question.** Is $x \in S$?

   Most of the problems that we look at will be decision problems.

2. A *functional problem* is a problem where the input is a string (over the *input alphabet*) and the output is a string (over the *output alphabet*).

4.5 Types

We will deal with several different types of things. It is essential that you know what type of thing each of your variables (or, in general, names) is.

Adjectives or other terms that we define can only be applied to certain types of things. For example, it makes sense to talk about the cardinality of a set, but not the cardinality of a number. The following is a list of some of the types of things that we will use.

Type	**Meaning**
boolean	A boolean value is either true or false. It might equally well be either 1 or 0, or either yes or no.
symbol	A symbol is a member of some alphabet.
string	A string is a (possibly empty) finite sequence of symbols
language	A language is a set of strings. We can think of a language as a decision problem.
function	Our functions will usually either take a string and yield a boolean value or will take a string and yield a string.
set of languages	A set of languages is called a *class*. We think of a language as a decision problem, and we will identify classes of decision problems that can be solved in particular ways.

5 Finite-State Machines and Regular Languages

This section looks at a simple model of computation for solving decision problems: a finite-state machine. A finite-state machine is also called a finite-state automaton (ah-TOM-a-tawn, plural automata), and the finite-state machines that we look at here are called *deterministic finite automata*, or DFAs.

Finite-state machines of a variety of flavors occur in other settings. For example, the processor that is at the heart of a computer is modeled as a finite-state machine. Compilers for programming languages use finite-state machines in their design.

BIG IDEA: We can define a model of computation, finite-state machines, in a precise and economical way.

5.1 Intuitive Idea of a DFA

Figure 5.1 shows a diagram, called a *transition diagram*, of DFA M_1. Each circle or double-circle is called a *state*. One of the states, marked by an arrow, is called the *start state*. A state with a double circle is called an *accepting state* and a state with a single circle is called a *rejecting state*.

The arrows between states are called *transitions*, and each transition is labeled by a member of the DFA's alphabet Σ (set $\{a, b\}$ for M_1).

Important. For each state q and each member c of Σ, there must be exactly one transition going out of q labeled c.

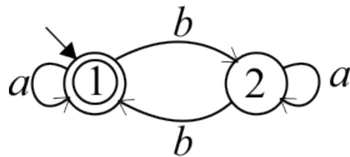


Figure 5.1. Transition diagram of DFA M_1 that recognizes language $\{s \in \{a, b\}^* \mid s \text{ has an even number of } b\text{'s}\}$. There are two states. State 1 is the start state. State 1 is an accepting state and state 2 is a rejecting state.

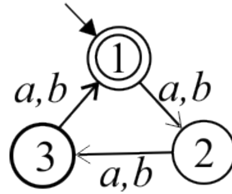


Figure 5.2. Transition diagram of DFA M_2, which accepts strings whose length is divisible by 3.

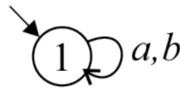


Figure 5.3. Transition diagram of DFA M_3, which rejects all strings.

A DFA is used to recognize a language (a decision problem). To "run" a DFA on string s, start in the start state. Read each character, and follow the transition labeled by that character to the next state. On input "$aabab$", M_1 starts in state 1, then hits states 1, 1, 2, 2, 1, ending in state 1.

The end state determines whether the DFA accepts or rejects the string. Since state 1 is an accepting state, M_1 accepts "$aabab$". It should be easy to see that M_1 accepts strings with an even number of b's and rejects strings with an odd number of b's.

A DFA M with alphabet Σ *recognizes* the set

$$L(M) = \{s \mid s \in \Sigma^* \text{ and } M \text{ accepts } s\}.$$

For example, $L(M_1) = \{s \mid s \in \{a, b\}^* \text{ and } s \text{ has an even number of } b\text{'s}\}$. Figures 5.2 and 5.3 show two finite-state machines M_2 and M_3 with alphabet $\{a, b\}$ where

$$\begin{aligned} L(M_2) &= \{s \mid \ |s| \text{ is divisible by } 3\} \\ L(M_3) &= \{\} \end{aligned}$$

5.2 Designing DFAs

BIG IDEA: A finite-state machine is best understood in terms of the set of strings that reach each state.

There is a simple and versatile way to design a DFA to recognize a selected language L. Associate with each state q the set of strings $\text{Set}(q)$ that end on state q. For example, in machine M_2,

$$\begin{aligned}\text{Set}(0) &= \{s \mid |s| \equiv 0 \pmod 3\}\\ \text{Set}(1) &= \{s \mid |s| \equiv 1 \pmod 3\}\\ \text{Set}(2) &= \{s \mid |s| \equiv 2 \pmod 3\}\end{aligned}$$

Your goals in designing a DFA that recognizes language L are:

(a) Start by deciding what the states will be and what $\text{Set}(q)$ will be for each state. Make sure that, for each state q, either $\text{Set}(q) \subseteq L$ (so that q is an accepting state) or $\text{Set}(q) \subseteq \overline{L}$, (so that q is a rejecting state).

(b) Draw transitions so that, if $x \in \text{Set}(q)$ and there is a transition from state q to state q' labeled a, then $x \cdot a \in \text{Set}(q')$.

5.2.1 Example: Even Binary Numbers

Figure 5.4 shows a DFA with alphabet {0,1} that accepts all even binary numbers. For example, it accepts `"10010"` and rejects `"1101"`. $\text{Set}(0) = \{s \in \{0,1\}^* \mid s \text{ is an even binary number}\}$ and $\text{Set}(1) = \{s \in \{0,1\}^* \mid s \text{ is an odd binary number}\}$. The transitions are obvious: adding a 0 to the end of any binary number makes the number even, and adding a 1 to the end makes the number odd.

5.2.2 A DFA Recognizing Binary Numbers that are Divisible by 3

Figure 5.5 shows a DFA that recognizes binary numbers that are divisible by 3. For example, it accepts `"1001"` and `"1100"`, since `"1001"` is the binary

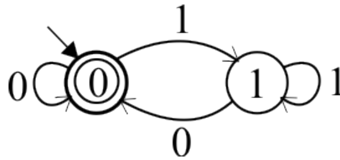


Figure 5.4. A DFA that recognizes even binary numbers. An empty string is treated as 0.

representation of 9 and "1100" is the binary represention of 12. But it rejects "100", the binary representation of 4.

Thinking of binary strings as representing numbers,

$$\begin{aligned}\text{Set}(0) &= \{n \mid n \equiv 0 \pmod 3\}\\ \text{Set}(1) &= \{n \mid n \equiv 1 \pmod 3\}\\ \text{Set}(2) &= \{n \mid n \equiv 2 \pmod 3\}\end{aligned}$$

Suppose that m is a binary number that is divisible by 3. Adding a 0 to the end doubles the number, so $m \cdot 0$ is also divisible by 3. (Adding 0 to then end of "1001" (9_{10}) yields "10010" (18_{10}).) Adding a 1 to m doubles m and adds 1. But modular arithmetic tells us that

$$\begin{aligned} m \equiv 0 \pmod 3 &\rightarrow 2m \equiv 0 \qquad \pmod 3\\ &\rightarrow 2m+1 \equiv 1 \quad \pmod 3\end{aligned}$$

so there is a transition from state 0 to state 1 on symbol 1. You can work out the other transitions.

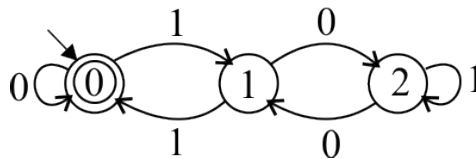


Figure 5.5. A DFA recognizing binary numbers that are divisible by 3. An empty string is treated as 0.

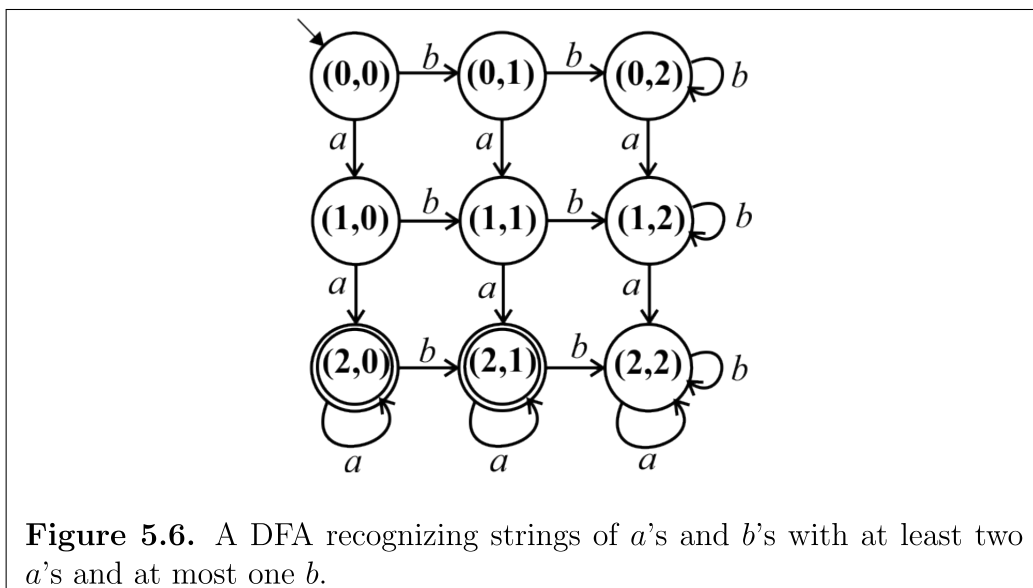


Figure 5.6. A DFA recognizing strings of a's and b's with at least two a's and at most one b.

5.2.3 Strings Containing at Least Two a's and at Most One b.

Figure 5.6 shows a DFA that regognizes language

$$\{w \in \{a, b\}^* \mid w \text{ contains at least two } a\text{'s and at most one } b\}.$$

The idea is to keep track of the number of a's (up to a maximum of 2) and the number of bs (up to a maximum of 2). That suggests that we need nine states: $(0, 0)$, $(0, 1)$, $(0, 2)$, $(1, 0)$, $(1, 1)$, $(1, 2)$, $(2, 0)$, $(2, 1)$ and $(2, 2)$, where the first number is the count of a's and the second the count of b's, and 2 means at least 2. The accepting states and transitions should be obvious.

5.3 Definition of a DFA and the Class of Regular Languages

The introduction above only shows transition diagrams, and does not adequately say exactly what a DFA is and how to determine the language that it recognizes. This section corrects that with a careful definition of both. The first definition says what a DFA is without saying what it means to run that machine on a string. It is, in a sense, just the syntax of a DFA.

5.3.1 Definition of a DFA

Definition 5.1. A *deterministic finite-state machine* is a 5-tuple (Σ, Q, q_0, F, δ). That is, the DFA is described by those five parts.

- Σ is the machine's alphabet.
- Q is a finite nonempty set whose members are called *states*.
- $q_0 \in Q$ is called the *start state*.
- $F \subseteq Q$ is the set of *accepting states*. (All members of $Q-F$ are *rejecting states*.)
- $\delta : Q \times \Sigma \rightarrow Q$ is called the *transition function*.

From state q, if you read symbol a, you go to state $\delta(q, a)$. Notice that, because δ is a function, there must be exactly one state to go to from state q upon reading symbol a.

5.3.2 When Does DFA M Accept String s?

Consider a DFA $M = (\Sigma, Q, q_0, F, \delta)$.

Definition 5.2. If $q \in Q$ and $x \in \Sigma^*$, then $q\,\mathbf{:}\,x$ is defined inductively as follows.

1. $q\,\mathbf{:}\,\varepsilon = q$.
2. If $x = cy$ where $c \in \Sigma$ and $y \in \Sigma^*$ then $q\,\mathbf{:}\,x = \delta(q, c)\,\mathbf{:}\,y$.

The idea is that $q\,\mathbf{:}\,x$ is the state that M reaches if it starts in state q and reads string x.

Every DFA M has a language $L(M)$ that it recognizes, and the following definition says what that is.

Definition 5.3. $L(M) = \{x \in \Sigma^* \mid q_0\,\mathbf{:}\,x \in F\}$.

That is, M accepts string x if M reaches an accepting state when it is run on x starting in the start state, q_0.

5.3.3 The Class of Regular Languages

Definition 5.4. Language A is *regular* if there exists a DFA M such that $L(M) = A$.

We have see a few regular languages above, including $\{\}$ and the set of binary numbers that are divisible by 3.

5.4 A Theorem about $q : x$

Notation $q : x$ satisfies a certain kind of associativity.

Theorem 5.5. $(q : x) : y = q : (xy)$.

Proof. The proof is by induction of the length of x. It suffices to

(a) show that $(q : x) : y = q : (xy)$ for all q and y when $|x| = 0$, and

(b) show that $(q : x) : y = q : (xy)$ for an arbitrary nonempty string x, under the assumption (called the *induction hypothesis*) that $(r : z) : y = r : (zy)$ for any state r, string y and string z that is shorter than x.

Case 1 ($|x| = 0$). That is, $x = \varepsilon$. By definition, $q : \varepsilon = q$. So

$$\begin{aligned} (q : x) : y &= q : y \\ &= q : (xy) \end{aligned}$$

because, when $x = \varepsilon$, $xy = y$.

Case 2 ($|x| > 0$). A nonempty string x can be broken into $x = cz$ where c is the first symbol of x and z is the rest.

$$\begin{aligned} (q : x) : y &= (q : (cz)) : y \\ &= (\delta(q, c) : z) : y && \text{by the definition of } q : (cz) \\ &= \delta(q, c) : (zy) && \text{by the induction hypothesis} \\ &= q : (czy) && \text{by the definition of } q : (czy) \\ &= q : (xy) && \text{since } x = cz \end{aligned}$$

5.5 Closure Results

A *closure* result tells you that a certain operation does not take you out of a certain set. For example, $\mathcal{Z}$ is *closed under addition* because the sum of two integers is an integer. $\mathcal{Z}$ is also *closed under multiplication*. But $\mathcal{Z}$ is not closed under division, since $1/2$ is not an integer.

The class of regular languages possesses some useful closure results.

Definition 5.6. Suppose that $A \subseteq \Sigma^*$ is a language. The complement $\overline{A}$ of A is $\Sigma^* - A$.

Theorem 5.7. The class of regular languages is closed under complementation. That is, if A is a regular language then $\overline{A}$ is also a regular language. Put another way, for every DFA M, there is another DFA M' where $L(M') = \overline{L(M)}$. Moreover, there is an algorithm that, given M, finds M'. That is, the proof is constructive.

Proof. Suppose that $M = (\Sigma, Q, q_0, F, \delta)$. Then $M' = (\Sigma, Q, q_0, Q - F, \delta)$. That is, simply convert each accepting state to a rejecting state and each rejecting state to an accepting state.

$\diamondsuit$

Theorem 5.8. The class of regular languages is closed under intersection. That is, if A and B are regular languages then $A \cap B$ is also a regular language. Put another way, suppose M_1 and M_2 are DFAs with the same alphabet Σ. There is a DFA M' so that $L(M') = L(M_1) \cap L(M_2)$. That is, M' accepts x if and only if both M_1 and M_2 accept x. Moreover, there is an algorithm that takes parameters M_1 and M_2 and produces M'.

Proof. The idea is to make M' simulate M_1 and M_2 at the same time. For that, we want a state of M' to be an ordered pair holding a state of M_1 and a state M_2. Recall that the cross product $A \times B$ of two sets A and B is $\{(a, b) \mid a \in A \wedge b \in B\}$.

Suppose that $M_1 = (\Sigma, Q_1, q_{0,1}, F_1, \delta_1)$. and $M_2 = (\Sigma, Q_2, q_{0,2}, F_2, \delta_2)$. Then $M' = (\Sigma, Q', q_0', F', \delta')$ where

$$\begin{aligned} Q' &= Q_1 \times Q_2 \\ q_0' &= (q_{0,1}, q_{0,2}) \end{aligned}$$

$$\begin{aligned} F' &= F_1 \times F_2 \\ \delta'((r,s),a) &= (\delta_1(r,a), \delta_2(s,a)) \end{aligned}$$

State (r,s) of M' indicates that M_1 is in state r and M_2 is in state s. Transition function δ' runs M_1 and M_2 each one step separately. Notice that the set F' of accepting states of M' contains all states (r,s) where r is an accepting state of M_1 and s is an accepting state of M_2. So M' accepts x if and only if both M_1 and M_2 accept x.

♢

Theorem 5.9. The class of regular languages is closed under union. That is, if A and B are regular languages then $A \cup B$ is also a regular language.

Proof. By DeMorgan's laws for sets,

$$A \cup B = \overline{\overline{A} \cap \overline{B}}.$$

But we already know that the class of regular languages is closed under complementation and intersection.

♢

6 Nonregular Languages

In this section we see how to prove that a language is not regular.

BIG IDEA: You *can* prove a negative.

6.1 A Motivating Example

Notation a^n means a string of n consecutive a's. For example, a^1 = "a", a^2 = "aa" and a^3 = "aaa". It is easy to design a finite-state machine that solves language

$$L_1 = \{a^m b^n \mid m > 0 \text{ and } n > 0\}.$$

A string s is in L_1 if and only if s consists of some positive number of a's followed by a positive number of b's. But suppose that

$$L_2 = \{a^n b^n \mid n > 0\}.$$

Notice that a string s is in L_2 if and only if s consists of some positive number of a's followed by *the same number* of b's. L_2 = {"ab", "$aabb$", "$aaabbb$", ...}.

Suppose that you want to design a finite state machine M where $L(M) = L_2$. What information does M need to remember? What if M reads a string of n a's and the next symbol is a b? M must remember n. If it doesn't, then how will M be able to check whether there are exactly n b's?

So it seems that M must have a state remembering that it has read exactly 1 a, another state remembering that it has read exactly 2 a's, another remembering that it has read exactly 3 a's, etc., without any limit. But that requires infinitely many states!

Can we conclude that L_2 is not regular? Be careful! Many incorrect "proofs" have been proposed that follow the rough outline: "I can only see one way to solve this problem. That way does not work. Therefore, this problem is unsolvable." That is nonsense. What if you missed an idea? We need a more careful proof.

6.2 A Proof Technique

The above idea about why L_2 is not regular is sound, but it needs to be presented more carefully. This section illustrates a way to show that a language is not regular (if it really isn't regular), using language L_2 as an example.

Theorem 6.1. L_2 is not regular.

Proof.

1. The proof is by contradiction. Suppose that L_2 is regular. We need to derive a contradiction by proving that **F** is true.

Know:	L_2 is regular.
Goal:	**F**.

2. Our knowlede uses term *regular*. By definition, L_2 is regular if an only if there is a DFA M where $L(M) = L_2$.

Know:	There exists a DFA M where $L(M) = L_2$.
Goal:	**F**.

3. When you know that there exists something with a particular property, ask someone else to give you such a thing. So let's ask for M, and suppose the start state of M is q_0.

Known variables:	M, q_0
Know:	$L(M) = L_2$.
Know:	q_0 is the start state of M.
Goal:	**F**.

4. This kind of proof involves a clever idea, and here it is. Our intuitive reasoning above looked at the state that M reaches after reading each of a^1, a^2, a^3, etc. So let's think about those states. In fact, since we have M in hand, we can do an experiment where we run M on each of a^1, a^2, a^3, a^4, etc. For each one, let's write the state that M reaches. That gives a table that *might* start out looking like this.

Input x	**State** $q_0 : x$ **reached**
"a"	2
"aa"	6
"aaa"	3
"$aaaa$"	9
...	...

But M only has finitely many states. By the pigeonhole principle, as you expand the table for longer and longer strings of a's, there must come a point where a state is repeated. Suppose that strings a^i and a^k take M to the same state q, where $i < k$.

Input x	**State** $q_0 : x$ **reached**
...	...
a^i	q
...	...
a^k	q
...	...

The experiment shows that $q_0 : a^i = q_0 : a^k = q$.

Known variables:	M, q_0, q, i, k
Know:	$L(M) = L_2$.
Know:	q_0 is the start state of M.
Know:	$q_0 : a^i = q$.
Know:	$q_0 : a^k = q$.
Know:	$i < k$.
Goal:	**F**.

5. Now comes a second clever trick. We have seen that M forgets the difference between a^i and a^k, since the only thing M can remember is the state that it is in. What if i b's come next? On input $a^i b^i$, M should answer yes. But on input $a^k b^i$, M should answer no.

Define $q' = q : b^i$. Recalling that $q = q_0 : a^i$ and $q = q_0 : a^k$,

$$\begin{aligned} q' &= q : b^i \\ &= (q_0 : a^i) : b^i \\ &= q_0 : a^i b^i && \text{(by Theorem 5.5)} \\ q' &= q : b^i \\ &= (q_0 : a^k) : b^i \\ &= q_0 : a^k b^i && \text{(by Theorem 5.5)} \end{aligned}$$

So M reaches the same state q' on input $a^i b^i$ as on input $a^k b^i$.

Suppose that q' is an accepting state. Then M correctly accepts $a^i b^i$ but incorrectly accepts $a^k b^i$.

Suppose that q' is a rejecting state. Then M correctly rejects $a^k b^i$ but incorrectly rejects $a^i b^i$.

No matter what, M does not correctly solve language L_2.

Known variables:	M
Know:	$L(M) = L_2$.
Know:	$L(M) \neq L_2$.
Goal:	**F**.

6. That gives us the contradiction: $(L(M) = L_2) \wedge (L(M) \neq L_2)$ is equivalent to **F**.

$\diamondsuit$

The above proof is actually quite constructive. Suppose that Archibald says he can produce a DFA M that solves L_2. Ask Archibald to give you M. Perform the above experiment. You find a string on which M gets the wrong answer. Sending that string to Archibald provides him with an irrefutable reason to believe that he was mistaken, and that M does not solve L_2.

6.3 Another Example

Suppose

$$L_3 = \{a^n \mid n \text{ is a perfect square}\}.$$

Theorem 6.2. L_3 is not regular.

Proof.

1. The proof is by contradiction. Suppose that L_3 is regular. We need to derive a contradiction by proving that **F** is true.

Know:	L_3 is regular.
Goal:	**F**.

2. By definition, L_3 is regular if an only if there is a DFA M where $L(M) = L_3$.

Know:	There exists a DFA M where $L(M) = L_3$.
Goal:	**F**.

3. Ask someone else to give you a DFA M where $L(M) = L_3$. Suppose the start state of M is q_0.

Known variables:	M, q_0
Know:	$L(M) = L_3$.
Know:	q_0 is the start state of M.
Goal:	**F**.

4. To employ the first clever idea, we need to find an infinite sequence of strings to try M on. The requirement is that M cannot afford to forget the difference between any two of those infinitely many strings; it needs to stop in a different state for each of them. Finding that sequence is the part of this kind of proof that requires the most thought.

   A sequence of strings that does the job is a^1, a^4, a^9, a^{16}, etc.; that is, run M on sequences of a's of lengths 1^2, 2^2, 3^2, 4^2, etc. We have M in hand, and we can do an experiment where we run M on each of those strings. The table *might* start out looking like this.

Input x	**State** $q_0 : x$ **reached**
a^{1^2}	8
a^{2^2}	1
a^{3^2}	14
a^{4^2}	6
...	...

Since M only has finitely many states, but there are infinitely many strings in the sequence, the right-hand column must eventually contain a repetition. Suppose that inputs a^{i^2} and a^{k^2} stop on the same state, q.

Input x	**State** $q_0 : x$ **reached**
...	...
a^{i^2}	q
...	...
a^{k^2}	q
...	...

Known variables:	M, q_0, q, i, k
Know:	$L(M) = L_2$.
Know:	q_0 is the start state of M.
Know:	$q_0 : a^{i^2} = q$.
Know:	$q_0 : a^{k^2} = q$.
Know:	$i < k$.
Goal:	**F**.

5. For the second clever trick, we must show that the first clever trick was chosen correctly. We have seen that M forgets the difference between a^{i^2} and a^{k^2}, since the only thing M can remember is the state that it is in. Our goal is to find *one* string r where M should accept $a^{i^2}r$ but

M should reject $a^{k^2}r$. A string r that does the job is $r = a^{2i+1}$. Notice that

$$\begin{aligned} a^{i^2}r &= a^{i^2}a^{2i+1} \\ &= a^{i^2+2i+1} \\ &= a^{(i+1)^2} \end{aligned}$$

So $a^{i^2}r \in L_3$. But

$$\begin{aligned} a^{k^2}r &= a^{k^2}a^{2i+1} \\ &= a^{k^2+2i+1} \end{aligned}$$

But $i < k$, so

$$\begin{aligned} k^2 &< k^2 + 2i + 1 \\ &< k^2 + 2k + 1 \\ &= (k+1)^2 \end{aligned}$$

Since there are no perfect squares between k^2 and $(k+1)^2$, $k^2 + 2i + 1$ cannot be a perfect square. That means $a^{k^2}r \notin L_3$.

Recall that M stops in the same state, q, on input a^{i^2} as on input a^{k^2}. Therefore, it stops on the same state $q' = q : r$ on input $a^{i^2}r$ as on input $a^{k^2}r$.

If q' is an accepting state, then M correctly accepts $a^{i^2}r$ but incorrectly accepts $a^{k^2}r$.

If q' is an rejecting state, then M correctly rejects $a^{k^2}r$ but incorrectly rejects $a^{i^2}r$.

No matter what, there is an input on which M gives the wrong answer. So $L(M) \neq L_3$.

Known variables:	M
Know:	$L(M) = L_3$.
Know:	$L(M) \neq L_3$.
Goal:	**F**.

6. That gives us the contradiction: $(L(M) = L_3) \wedge (L(M) \neq L_3)$ is equivalent to **F**.

$\diamondsuit$

6.4 Yet Another Example

Suppose

$$L_4 = \{ww \mid w \in \{a, b\}^*\}.$$

Strings in L_4 include "aa", "$abab$", "$aabbbaabbb$" and "$bbaabbaa$", among infinitely many others.

Theorem 6.3. L_4 is not regular.

Proof.

1. As before, the proof is by contradiction. Suppose that L_4 is regular. We need to derive a contradiction by proving that **F** is true.

Know:	L_4 is regular.
Goal:	**F**.

2. By definition, L_4 is regular if an only if there is a DFA M where $L(M) = L_4$.

Know:	There exists a DFA M where $L(M) = L_4$.
Goal:	**F**.

3. Ask someone else to provide us with a DFA M where $L(M) = L_4$. Suppose the start state of M is q_0.

Known variables:	M, q_0
Know:	$L(M) = L_4$.
Know:	q_0 is the start state of M.
Goal:	**F**.

4. We need to find an infinite sequence of strings to try M on, where M cannot afford to forget the difference between any two of those strings. A sequence that works is a^1b, a^2b, a^3b, etc. Let's try running M on those strings and write down the state that M reaches for each of them. The experiment *might* yield the following.

Input x	**State** $q_0 : x$ **reached**
"ab"	1
"aab"	2
"$aaab$"	3
"$aaaab$"	4
...	...

But M only has finitely many states. As you expand the table for longer and longer strings, there must come a point where a state is repeated. Suppose that strings a^ib and a^kb take M to the same state q.

Input x	**State** $q_0 : x$ **reached**
...	...
a^ib	q
...	...
a^kb	q
...	...

The experiment shows that $q_0 : a^ib = q_0 : a^kb = q$.

Known variables:	M, q_0, q, i, k
Know:	$L(M) = L_4$.
Know:	q_0 is the start state of M.
Know:	$q_0 : a^ib = q$.
Know:	$q_0 : a^kb = q$.
Know:	$i < k$.
Goal:	**F**.

5. M forgets the difference between $a^i b$ and $a^k b$, since the only thing M can remember is the state that it is in. What if string $r = a^i b$ comes next? On input $a^i b a^i b$, M should answer yes, since $a^i b a^i b = ww$ where $w = a^i b$. But on input $a^k b a^i b$, M should answer no, since there does not exist any string w where $a^k b a^i b = ww$. But M reaches the same state $q' = q : r$ on input $a^i b a^i b$ as on input $a^k b a^i b$. If q' is an accepting state, then M incorrectly accepts $a^k b a^i b$. If q' is a rejecting state, then M incorrectly rejects $a^i b a^i b$. So M does not solve L_4.

Known variables:	M
Know:	$L(M) = L_4$.
Know:	$L(M) \neq L_4$.
Goal:	**F**.

6. That gives us the contradiction.

♢

I have done these proofs in with a lot of detail. Here is the the previous proof done in a more typical way.

Theorem 6.4. L_4 is not regular.

Proof. Suppose L_4 is regular. Let M by a DFA that solves L_4.

Imagine doing an experiment where you run M on strings $a^1 b$, $a^2 b$, $a^3 b$, and so on. Because M has finitely many states, eventually values i and k must be found, with $i < k$, where $a^i b$ and $a^k b$ take M to the same state q.

Now imagine running M on inputs $a^i br$ and $a^k br$ where $r = a^i b$. Since M reaches the same state on inputs $a^i b$ and $a^k b$, M must also reach the same state on inputs $a^i br$ and $a^k br$. So M accepts both $a^i br$ and $a^k br$ or it rejects both. But that means that M does not solve L_4, since $a^i b a^i b \in L$ and $a^k b a^i b \notin L$. That is a contradiction.

♢

6.5 A Common Mistake

Suppose $L_5 = \{a^n a^n \mid n > 0\}$. Let's try to prove the following.

Claim. L_5 is not regular.

"Proof."

1. The proof is by contradiction. Suppose that L_5 is regular. We need to derive a contradiction by proving that **F** is true.

Know:	L_5 is regular.
Goal:	**F**.

2. By definition, L_5 is regular if and only if there is a DFA M where $L(M) = L_5$.

Know:	There exists a DFA M where $L(M) = L_5$.
Goal:	**F**.

3. Ask someone else to give such you a DFA M where $L(M) = L_5$, and suppose the start state of M is q_0.

Known variables:	M, q_0
Know:	$L(M) = L_5$.
Know:	q_0 is the start state of M.
Goal:	**F**.

4. Do an experiment using M. Run M on strings a^1, a^2, a^3, etc. and record the state reached for each string. Continue until a state q has been written twice, which must happen because M has finitely many states.

Input x	**State** $q_0 : x$ **reached**
...	...
a^i	q
...	...
a^k	q
...	...

Known variables:	M, q_0, q, i, k
Know:	$L(M) = L_2$
Know:	q_0 is the start state of M
Know:	$q_0 : a^i = q$
Know:	$q_0 : a^k = q$
Know:	$i < k$
Goal:	**F**

5. Now we need to find a string r so that $a^i r \in L_5$ but $a^k r \notin L_5$. Choose $r = a^i$.

   Notice that $a^i r = a^i a^i$ and that is in L_5 from the definition of L_5.

   Notice that $a^k r = a^k a^i$. But that does not have the form $a^n a^n$ so $a^k r \notin L_5$.

   As before, that leads to a contradiction.

◇

But that "proof" cannot be correct. $\{a^n a^n \mid n > 0\} = \{a^{2n} \mid n > 0\}$. So L_5 is the set of all strings of a's whose length is even, and that is a regular language. Where did the proof go wrong?

The incorrect proof states that $a^k a^i$ does not have the form $a^n a^n$. Suppose $k = 4$ and $i = 2$. Then $a^k a^i = a^4 a^2 = a^6 = a^3 a^3$. In fact, as long as $i + k$ is even, $a^k a^i$ *does* have the form $a^n a^n$, where $n = (i + k)/2$.

Can you insist that $i + k$ is odd? Clearly not. The claim is false. It is pointless to try to modify the proof since the claim is false.

6.6 Be Careful Not to Be Sloppy

Having seen a few proofs like the above, all using similar ideas, it is easy to get the idea that it is not necessary to write out all of the details, and instead to skip directly to step 5. But step 4 says what the experiment is; that is, what is the infinite sequence of strings to run M on? If you don't say what the experiment is, you will find yourself making inconsistent statements about that experiment.

There is an easy way to avoid that. Don't skip the details. Write them down and check that what you have written is sensible. Look at an example. (We found that the above incorrect proof was not right by looking at the example $i = 2$ and $k = 4$.)

You don't need to write out tables of known things and goals, as in our very detailed proofs. Use the typical (shorter) proof style. But don't expect a person who reads your proof to fill in important details, such as the nature of the experiment, or why one string is in $L(M)$ while the other is not.

7 Regular Expressions

This section introduces regular expressions. A regular expression describes a set of strings. We will see that the class of languages that can be decribed by regular expressions is the same as the class of regular languages (those languages that can be solved by a finite-state machine).

Regular expressions are of practical value; some programming language libraries provide tools for doing searches based on regular expressions. Some text editors and console command languages similarly provide tools for searching for something that is a member of the set described by a given regular expression.

7.1 Regular Operations

The *regular operations* are operations on languages; they take one or more languages and yield another language, in the same sense that operator + takes two numbers and yields another number. The first regular operation is union ($A \cup B$), which we have already seen. The remaining two regular operations are concatenation and Kleene closure.

Definition 7.1. The *concatenation* $A \cdot B$ of languages A and B is defined by

$$A \cdot B = \{xy \mid x \in A \text{ and } y \in B\}.$$

That is, $A \cdot B$ is the set of all strings that can be formed by writing a member of A followed by a member of B. For example, $\{\texttt{"}aa\texttt{"}, \texttt{"}ccb\texttt{"}\} \cdot \{\texttt{"}abc\texttt{"}, \texttt{"}bb\texttt{"}\} = \{\texttt{"}aaabc\texttt{"}, \texttt{"}aabb\texttt{"}, \texttt{"}ccbabc\texttt{"}, \texttt{"}ccbbb\texttt{"}\}$.

Definition 7.2. The *Kleene closure* A^* of language A is defined by

$$A^* = \{x_1 x_2 \cdots x_n \mid n \geq 0 \text{ and } x_i \in A \text{ for } i = 1, \ldots, n\}.$$

If $A = \{\texttt{"}a\texttt{"}, \texttt{"}bcb\texttt{"}\}$ then $A^* = \{\varepsilon, \texttt{"}a\texttt{"}, \texttt{"}bcb\texttt{"}, \texttt{"}aa\texttt{"}, \texttt{"}abcb\texttt{"}, \texttt{"}bcba\texttt{"}, \texttt{"}bcbbcb\texttt{"}, \ldots\}$. A^* contains the empty string and all strings that can be formed by concatenating members of A together. Notice that $\{\}^* = \{\varepsilon\}$.

Language L is *closed under concatenation* if, whenever x and y are both in L, xy is also in L. Another way to define the Kleene closure of A is as the smallest set of strings that is closed under concatenation and that contains ε all members of A.

7.2 Regular Expressions

A regular expression e over alphabet Σ is an expression whose value is a language $L(e)$ over Σ. Regular expressions have the following forms.

1. Symbol $\emptyset$ is a regular expression. $L(\emptyset) = \{\}$.
2. A symbol $a \in \Sigma$ is a regular expression. $L(a) = \{"a"\}$.
3. If A and B are regular expressions, then:
   (a) $A \cup B$ is a regular expression. $L(A \cup B) = L(A) \cup L(B)$.
   (b) AB is a regular expression. $L(AB) = L(A) \cdot L(B)$.
   (c) A^* is a regular expression. $L(A^*) = L(A)^*$.

Conventionally, * has highest precedence, followed by concatenation, with $\cup$ having lowest precedence. You can use parentheses to override precedence rules.

We put spaces in some regular expressions to make them more readable. They don't affect the meaning.

7.3 Examples of Regular Expressions

$(ab)^*$	Any string over alphabet $\{a, b\}$ that consists of ab repeated zero or more times. $\{\varepsilon$, "ab", "$abab$", "$ababab$", $\ldots\}$
a^*b^*	Any string over alphabet $\{a, b\}$ that consists of zero or more a's followed by zero or more b's: $\{\varepsilon$, "a", "b", "ab", "aab", "$aabb$", $\ldots\}$.
$(a \cup b)^*$	All strings over alphabet $\{a, b\}$.
$(a \cup b)^* a (a \cup b)$	All strings over alphabet $\{a, b\}$ whose next-to-last symbol is a.
$(a \cup b)^* aabb (a \cup b)^*$	All strings over alphabet $\{a, b\}$ that have $aabb$ as a contiguous substring.
$b^*(ab^*a)^*b^*$	All strings over alphabet $\{a, b\}$ that have an even number of as.
$(0 \cup 1(01^*0)^*1)^*$	All binary numbers that are divisible by 3. (This one is difficult and is not obvious. Look at the DFA in Figure 5.1.1. Starting in state 0, what can the DFA read to get it back to state 0? Certainly, it can read a 0. It can also read a 1, taking it to state 1, then 01^*0 repeated any number of times, then a 1 to get it back to state 0. Those two, getting the DFA from state 0 back to state 0, can be repeated any number of times.

8 Equivalence of Regular Expressions and Finite-State Machines

> **BIG IDEA:** Sometimes, two very different definitions define the same thing.

We have defined the class of regular language using DFAs. We have also defined regular expressions. Finite-state machines and regular expressions could not be more different from one another. Amazingly, the class of languages that can be described by regular expressions is exactly the class of regular languages (the ones solvable by DFAs)! We will demonstrate the following two lemmas. (A lemma is a theorem that is proved as a step in proving a more important theorem.)

Lemma 8.1. If e is a regular expression then $L(e)$ is a regular language.

Lemma 8.2. If A is a regular language then there exists a regular expression e so that $L(e) = A$.

The following theorem follows immediately from the two lemmas.

Theorem 8.3. Language A is regular (solvable by a DFA) if and only if there exists a regular expression that describes language A.

Proofs of Lemmas 8.1 and 8.2 are long, and they involve defining new types of finite-state machines that are of an intermediate nature between regular expressions and DFAs.

8.1 Nondeterministic Finite-State Machines

Like a DFA, a *nondeterministic finite automaton*, or NFA, is a 5-tuple (Σ, Q, q_0, F, δ_N). An NFA differs from a DFA in the following two ways.

1. For each state q and each symbol $c \in \Sigma$, instead of giving a single state, the transition function $\delta_N(q, c)$ of gives a *set of states* that can be reached from q upon reading symbol c. For example, there can be several transitions from state 2 to other states, or there can be no transitions from state 2 to other states.

2. An NFA accepts string s just when there exists a path from the start state q_0 to an accepting state (a member of F), where the symbols along the path are labeled, in sequence, by the symbols in string s.

Figure 8.1 shows the transition diagram of an NFA that accepts all strings over alphabet $\{a, b\}$ that end on ab.

8.1.1 The Subset Construction

Theorem 8.4. For every NFA M there is an equivalent DFA M'. The two are equivalent in the sense that they accept the same language.

Proof Sketch. The proof is an algorithm, called the *subset construction*, that converts an NFA into an equivalent DFA. The idea is simple: make the DFA keep track of all states that the NFA could possibly be in. So each state of the DFA is a set of states of the NFA. Here are the main ideas of the subset construction.

1. If q_0 is the start state of the NFA, then the start state of the DFA is $\{q_0\}$. That is, the DFA starts out in a state where it can only be in state q_0.

2. A state of the DFA is an accepting state provided it contains at least one accepting state of the NFA.

3. Suppose that δ_N is the transition function of the NFA. The DFA has a transition from set s to set t labeled by symbol c provided

$$t = \bigcup_{q \in s} \delta_N(q, c).$$

Typically, not all of the sets of states are accessible from the start state. Figure 8.2 shows the DFA that is obtained from the NFA in Figure 8.1 by the subset construction. Inaccessible states are omitted.

◇

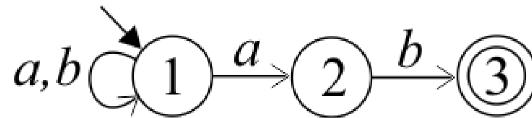


Figure 8.1. Transition diagram of an NFA that recognizes the set of all strings over alphabet $\{a, b\}$ that end on ab. For example, it accepts "ab", "$bbaab$" and "$aaaaaab$".
Notice that there are two transitions out of state 1 labeled by symbol a. There are no transitions leaving state 3.
To run an NFA, start in state q_0 and follow transitions, writing the symbol of each transition as you follow it. If it is possible to write down string s and stop at an accepting state, then s is in the language of the NFA.

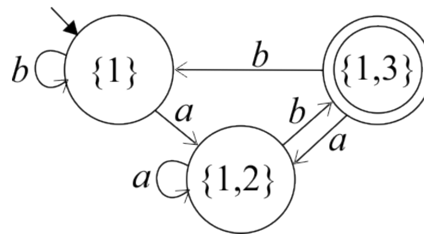


Figure 8.2. The DFA that is obtained from the NFA in Figure 8.1 using the subset construction. Each state of the DFA is a set of states of the NFA.

To build the DFA, start by creating start state $\{q_0\}$. Then create new states as they are needed.

For example, this DFA has a transition from state {1,2} to state {1,3} labeled b because there is a transition in the DFA from 1 to 1 labeled b and there is also a transition from 2 to 3 labeled b.

NFAs with ε-Transitions

An NFA with epsilon-transitions (an *NFA*$_\varepsilon$) is like an NFA, but it allows transitions labeled ε, called ε-transitions. You follow an ε-transition without reading a symbol.

It is easy to show that ε-transitions are not essential; you can get rid of them. That is, the following theorem is true. Its proof is left as an exercise. (Just add new non-epsilon-transitions that allow an NFA to reach all the same states as a particular NFA$_\varepsilon$.)

Theorem 8.5. For every NFA$_\varepsilon$ M, there is an equivalent NFA M' (without ε-transitions).

♢

There is an important property of NFAs with ε-transitions. You can convert any NFA$_\varepsilon$ to an equivalent one with only one accepting state. That is easy! Just add a new state (to be the sole accepting state), add an epsilon transition from each accepting state to the new accepting state, and finally make all of the states except the new accepting state rejecting states.

Converting a Regular Expression to an NFA with ε-Transitions

Now we are ready to show how to convert a regular expression to a DFA. The idea is to convert the regular expression to an NFA$_\varepsilon$, then to convert that to an ordinary NFA, then to convert the NFA to a DFA using the subset construction.

Theorem 8.6. For every regular expression, there is an equivalent NFA$_\varepsilon$.

Proof. The proof is by induction on the length of a regular expression. Refer to the definition of a regular language in Section 7. A regular expression has one of five different forms: $\emptyset$, c (where c is a symbol), $A \cup B$, AB and A^*.

It is obvious that there is an NFA for an empty set and for every singleton set that contains a string of length 1. Also, we already know that the set of regular language is closed under union; having shown that the languages of regular expressions A and B are regular, we can conclude that the language of $A \cup B$ is also regular by that closure result.

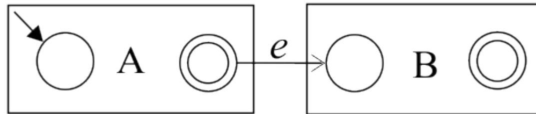


Figure 8.3. The regular languages are closed under concatenation. Suppose A and B are regular languages. Get an NFA$_\varepsilon$ for each of A and B, and ensure that the NFA$_\varepsilon$ has exactly one accepting state. Build an NFA$_\varepsilon$ as shown in the above diagram, connecting the NFA$_\varepsilon$ for A with that for B. (The ε-transition is labeled e.) Make the accepting state of A nonaccepting, and make only the start state of A be the start state of the combined machines.

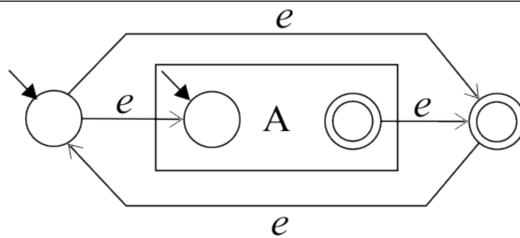


Figure 8.4. The regular languages are closed under Kleene closure. Suppose that A is a regular language. Get an NFA$_\varepsilon$ for A, and assume that it has exactly one accepting state. Build an NFA$_\varepsilon$ for A^* as shown in the diagram above.

All we need to do is to show that the regular languages are closed under concatenation and Kleene closure.

1. Suppose that A and B are regular languages. Then the concatenation AB is also regular. See Figure 8.3.

2. Suppose that A is regular language. Then the Kleene closure A^* of A is also regular. See Figure 8.4.

$\diamondsuit$

Now we are ready to prove Lemma 8.1.

Lemma 8.1. If e is a regular expression then $L(e)$ is a regular language.

Proof. Convert regular expression e to an NFA$_\varepsilon$, then to an NFA using Theorem 8.5, an finally to a DFA using the subset construction.

$\diamondsuit$

8.2 Converting a DFA to a Regular Expression

Now that we have shown how to convert a regular expression to an equivalent DFA (in a few steps), we need to show how to convert a DFA into an equivalent regular expression. For that, we need yet another type of finite-state machine.

A *generalized finite-state machine* (a GFA) has each transition labeled by a regular expression. The idea is that you can follow a transition while reading any member of the regular expression that labels the transition.

Theorem 8.7. For every DFA M there is an equivalent regular expression e.

Proof. Start with a DFA. It is a special case of an NFA$_\varepsilon$ that happens not to have any ε-transitions. Add a new accepting state and ε-transitions from all former accepting states so that there is exactly one accepting state. An NFA$_\varepsilon$ is a special case of a GFA, and we start with that GFA. (Replace ε by regular expression $\emptyset^*$.)

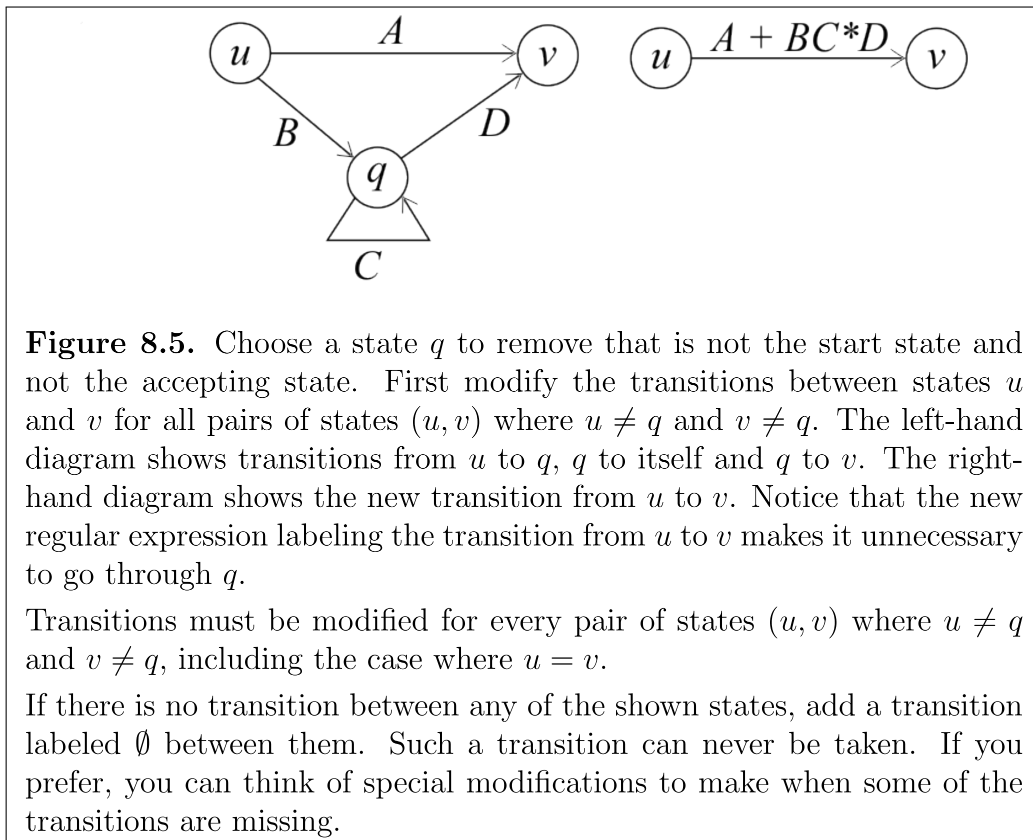


Figure 8.5. Choose a state q to remove that is not the start state and not the accepting state. First modify the transitions between states u and v for all pairs of states (u, v) where $u \neq q$ and $v \neq q$. The left-hand diagram shows transitions from u to q, q to itself and q to v. The right-hand diagram shows the new transition from u to v. Notice that the new regular expression labeling the transition from u to v makes it unnecessary to go through q.

Transitions must be modified for every pair of states (u, v) where $u \neq q$ and $v \neq q$, including the case where $u = v$.

If there is no transition between any of the shown states, add a transition labeled $\emptyset$ between them. Such a transition can never be taken. If you prefer, you can think of special modifications to make when some of the transitions are missing.

Now the idea is to remove states from the GFA one at a time, but not to remove the start or accepting state. Figure 8.5 shows how to remove state q. It is not the most efficient way to do that, but it gets the job done.

Eventually, you reach a GFA that has only two states, a start state and an accepting state, as shown in Figure 8.6. It is easy to convert that GFA into a regular expression.

♢

Lemma 8.2. If A is a regular language then there exists a regular expression e so that $L(e) = A$.

Proof. That is immediate from Theorem 8.7.

♢

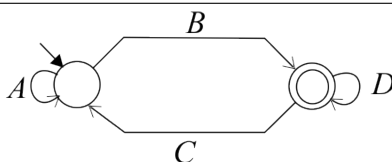


Figure 8.6. The final step in converting a GFA into a regular expression. The diagram shows the case where there are only two states left. Convert the GFA shown in the diagram into regular expression $A^*B(D \cup CA^*B)^*$. That captures all ways of getting from the start state to the accepting state.

9 Programs and Computability

9.1 Programs

With this section, we begin to look at what can be computed by general programs. But what is a general program?

A full definition of a general program is involved and takes us into an area, *automata theory*, that we will not explore in this course for lack of time. So let's settle for a less-than-rigorous defininition of a program.

Program "$\{p(x)$: *body*$\}$" is a program or function called p that performs actions indicated by *body*. In the body, a program says **return** r to indicate that the answer is r. Otherwise, the body is written in *psuedo-code* that you can imagine has been translated into your favorite programming language. We use indentation to show program structure.

Technically the input, or parameter, is always a string. But the input might be an integer, written in base 10. The input might also have more than one thing encoded in it. For example, input "(25,400)" describes an ordered pair of integers. So we will allow a program with more than one input, as in "$\{q(x, y)$: $\ldots\}$".

Program "$\{a(x_1x_2 \ldots x_n)$: $\ldots\}$" takes a parameter string $x =$ "$x_1x_2 \ldots x_n$"; in the body, x_i refers to the i-th character of x.

Some examples are shown later in this section.

9.1.1 A Program Is a String

We write a program in quotes because a program is a string. You create a program using a **text-**editor. That point is important for the study of computability. If there are string constants embedded inside the program, I will not write \" for the embedded quotes. There should be no confusion from that.

We refer to program "$\{p(x)$: $\ldots\}$" a p. Keep in mind that p is both a program and a string.

9.2 Computability

9.2.1 Computable Functions

Definition 9.1. For our purposes, an *algorithm* is a program that stops and produces an answer for every input. It is not allowed to loop forever, and is not allowed to stop without giving an answer.

Definition 9.2. Suppose that Σ and Γ are alphabets and $f : \Sigma^* \rightarrow \Gamma^*$ is a function. Program p *computes* function f provided, for every string $s \in \Sigma^*$, when p is run on input s, it eventually stops and returns string $f(s)$.

Definition 9.3. Function f is *computable* if there exists a program that computes f.

9.2.2 Computable Decision Problems

Definition 9.4. Suppose $A \subseteq \Sigma^*$ is a language over Σ^*. A program p *computes* A provided, for every string $s \in \Sigma^*$, when p is run on input s, it eventually stops and returns 1 if $s \in A$ and returns 0 if $s \notin A$.

If p computes A, we also say that p *solves* A, p *recognizes* A and that p *decides* A.

Definition 9.5. If p is an algorithm, define $L(p)$ to be the set of all strings on which program p stops and returns 1. We say that $L(p)$ is the language that p accepts.

Definition 9.6. Language A is *computable* provided there exists a program that computes A. Equivalently, A is computable if there exists a program p that stops on every input and where $L(p) = A$. Computable decision problems are also said to be *decidable*.

Note that computability is not defined in terms of what you or I are clever enough to do. A function or language is computable if *there exists* a program that computes it, regardless of whether any human is or will ever be able to find such a program.

9.2.3 The Church/Turing Thesis

Each programming language is a *model of computation*. Why can we ignore details like which programming language is chosen (within some limits) in the definition of a computable problem? Because every sufficiently general programming language can solve the same problems, as long as you take away restrictions on the amount of memory that the program can use. That observation is captured in the *Church/Turing Thesis*.

BIG IDEA: The Church/Turing Thesis: the class of computable problems is the same for all sufficiently general models of computation.

You hardly need much to achieve sufficiently general power. A common model of computation is a *Turing machine*, whose memory consists of an infinitely long tape that can store one symbol per cell, and that can only be read and written using a head that can move to the left and right over the tape. That model initially appears to be too simple, but it can solve all of the computational problems that are solvable by other models of computation.

9.2.4 The "Type" of Adjective *Computable*

A language can be computable. A function that takes a string and yields a string can be computable. A function that takes a number and yields a number can be computable.

But a program cannot be computable. It makes no sense to talk about a computable program. So please don't ever to that. Make sure that you know what type of thing you have.

9.3 Examples of Computable Decision Problems

It is easy to come up with computable decision problems.

Theorem 9.7. The empty set is computable.

Proof. Language $\{\}$ is thought of as the following decision problem.

Input. String x
Question. Is $x \in \{\}$?

Of course, the answer to the question is “no” regardless of what x is, and program "$\{e(x)$: return $0\}$" computes $\{\}$.

$\diamondsuit$

Theorem 9.8. Language $\{"b", "abb", "baba"\}$ is computable.

Proof. The following program t computes language $\{"b", "abb", "baba"\}$.

```
"{t(x):
   if x == "b"
      return 1
   else if x == "abb"
      return 1
   else if x == "baba"
      return 1
   else
      return 0
}"
```

$\diamondsuit$

You should be able to use the idea in the proof of Theorem 9.8 to prove the following.

Theorem 9.9. Every finite set is computable.

Theorem 9.10 shows there is a nonregular language that is computable. That should come as no surprise. General programs have much more power than finite-state machines.

Theorem 9.10. Language $\{a^n b^n \mid n > 0\}$ is computable.

Proof. Suppose that $\Sigma = \{a, b\}$. To compute $\{a^n b^n \mid n > 0\}$, it suffices to (1) check that there does not occur an a after a b, and (2) count the a's, count the b's, and check that the two counts are the same. The following program accomplishes that.

```
"{p(x_1 x_2 ... x_n):
  i = 1
  c_a = 0
  c_b = 0
  while i ≤ n and x_i == 'a'
    i = i + 1
    c_a = c_a + 1
  while i ≤ n and x_i == 'b'
    i = i + 1
    c_b = c_b + 1
  if i == n + 1 and c_a == c_b
    return 1
  else
    return 0
}"
```

♢

Theorem 9.11. Language $\{n \mid n \text{ is a prime integer}\}$ is computable.

Proof. The following program tells whether n is prime.

```
"{p(n):
  if n < 2
    return 0
  i = 2
  while i < n
    if n mod i == 0
      return 0
    i = i + 1
  return 1
}"
```

♢

9.4 Every Regular Language Is Computable

Theorem 9.12. Every regular language is computable.

Proof. Suppose that A is a regular language. That is, there exists a DFA M so that $L(M) = A$. Ask someone else to give us such a DFA $M = (\Sigma, Q, q_0, F, \delta)$. Here is a program $R(x)$ that solves A. It simply runs M on input x.

"{$R(x_1 x_2 \ldots x_n)$:
  $q = q_0$
  $i = 1$
  while $i \leq n$
    $q = \delta(q, x_i)$
    $i = i + 1$
  if $q \in F$
    return 1
  else
    return 0
}"

$\diamondsuit$

9.5 Computable Questions About DFAs

A program can take a DFA as an input. It is just a matter of encoding the DFA as a string. Suppose that $M = (\{a, b\}, \{1, 2, 3\}, 1, \{2, 3\}, \delta)$ where the transition function δ is as follows.

δ	a	b
1	1	2
2	3	1
3	1	1

A possible encoding of M as a string is

"{a,b}{1,2,3}1{2,3}(1,a:1)(1,b:2)(2,a:3),(2,b:1),(3,a:1)(3,b:1)".

Obviously, many different encodings would work.

9.5.1 Does M Accept x?

Definition 9.13. The *acceptance problem for DFAs* is the following decision problem.

> **Input.** A DFA M (encoded as a string) and a string x.
> **Question.** Does M accept x?

Theorem 9.14. The acceptance problem for DFAs is computable.

Proof. We have seen how to simulate a DFA M on input x. The only difference here is that M is encoded as a string. But that is not a problem; any experienced programmer can write a program that reads the encoding and pulls out all of the features of M.

◊

9.5.2 Does M Accept All Strings?

Let's look at a more difficult problem.

Definition 9.15. The *everything problem for DFAs* is the following decision problem.

> **Input.** A DFA M (encoded as a string) with alphabet Σ.
> **Question.** Does M accept all strings in Σ^*.

Solving the everything problem for DFAs might at first seem impossible. After all, there are infinitely many strings, and you can't check them all. But that is an illusion; it is actually quite easy to check whether M accepts all strings.

Theorem 9.16. The everything problem for DFAs is computable.

Proof. Suppose $M = (\Sigma, Q, q_0, F, \delta)$. Some DFAs have states that cannot be reached by any input string. M accepts all strings in Σ^* if every state that can be reached is an accepting state. The hardest part is determining the reachable states, and that is actually easy.

Assume that there is a *mark bit* associated with each state of M that a program can set to 0 or 1. (That is easy to arrange. If M's states are $\{1, \ldots, n\}$, all we need is an array of n boolean values to hold the mark bits.)

```
"{everything(M):
  // Mark all accessible states
  Set the mark bit of every state to 0.
  Set the mark bit of q0 to 1.
  changed = 1
  while changed == 1
    changed = 0
    for each state q of M
      if q's mark bit is 1
        for each symbol a in Σ
          r = δ(q, a)
          if r's mark bit is 0
            set r's mark bit to 1
            changed = 1
  // Check if there a marked rejecting state
  for each state q of M
    if q's mark bit is 1 and q ∉ F
      return 0
  return 1
}"
```

♢

9.5.3 Does M Accept No Strings?

Definition 9.17. The *emptiness problem for DFAs* is language $\{M \mid L(M) = \{\}\}$. That is, it is the following decision problem.

Input. DFA M (encoded as a string).
Question. Is it the case that M does not accept any strings?

Theorem 9.18. The emptiness problem for finite state machines is computable.

Proof. The proof is similar to the preceding proof, but the algorithm checks that each reachable state is a rejecting state.

♢

9.5.4 Is $L(M) \subseteq L(N)$?

Definition 9.19. The *subset problem for DFAs* is the following decision problem.

> **Input.** Two DFAs M_1 and M_2 (encoded as strings).
> **Question.** Is $L(M_1) \subseteq L(M_2)$? That is, is every string in $L(M_1)$ also in $L(M_2)$?

Once again, a shallow thought process leads one to conclude that the subset problem for DFAs is not computable, since there are infinitely many strings to check. A more careful look shows that it is computable.

Theorem 9.20. The subset problem for DFAs is computable.

Proof. We have seen, in Theorems 5.7 and 5.8, that the class of regular languages is closed under complementation and intersection. It is important that both theorems are proved by constructive proofs. That is,

1. There is an algorithm that, given a DFA M, produces DFA M' so that $L(M') = \overline{L(M)}$.
2. There is an algorithm that, given DFAs M_1 and M_2, produces DFA M' so that $L(M') = L(M_1) \cap L(M_2)$.

For any two sets A and B,

$$A \subseteq B \leftrightarrow A - B = \{\}.$$

But $A - B = A \cap \overline{B}$. The algorithm first builds DFA M_3 so that $L(M_3) = \overline{L(M_2)}$. Then it builds DFA M_4 so that

$$L(M_4) = L(M_1) \cap L(M_3) = L(M_1) \cap \overline{(L(M_2))} = L(M_1) - L(M_2).$$

So $L(M_1) \subseteq L(M_2) \leftrightarrow L(M_4) = \{\}$. But we have an algorithm (Theorem 9.7) to tell if $L(M_4) = \{\}$.

$\diamondsuit$

9.5.5 Are $L(M)$ and $L(N)$ the Same Language?

Definition 9.21. The *equivalence problem for DFAs* is the following decision problem.

Input. Two DFAs M_1 and M_2 (encoded as strings).
Question. Is $L(M_1) = L(M_2)$?

Theorem 9.22. The equivalence problem for DFAs is computable.

Proof. For any two sets A and B, by definition,

$$A = B \quad \leftrightarrow \quad A \subseteq B \wedge B \subseteq A.$$

It suffices to test each of $L(M_1) \subseteq L(M_2)$ and $L(M_2) \subseteq L(M_1)$ separately.

$\diamondsuit$

9.6 Computable Problems About Polynomials

Let's look at problems involving polynomials with integer coefficients, which we simply call polynomials. An input to such a problem might be $5x^2 - 2$ or $x^2 + 1$. A value of x that makes $5x^2 - 2 = 0$ is called a *zero* of polynomial $5x^2 - 2$.

Definition 9.23. The *real-zero problem* takes a polynomial p of variable x as input and asks whether there is a zero of p that belongs to $\mathcal{R}$, the set of real numbers.

For example, polynomial $x^5 - 2x^3 - 16$ has value 0 when $x = 2$, so it is a yes-input to the real-zero problem. Polynomial $4x^2 - 4x + 1$ is also a yes-input, since it has value 0 for $x = 1/2$.

9.6.1 Quadratic Single-Variable Polymomials

A naive first thought might be that the real-zero problem is not computable since an algorithm would have to try every possible number. But it should be clear that the zero problem is computable for quadratic polynomials. The quadratic formula tells you that equation $ax^2 + bx + c = 0$ has a real-valued solution if and only if $b^2 - 4ac \geq 0$.

9.6.2 Arbitrary Degree Single-Variable Polymomials

What if polynomials of x are allowed to have any degree? There are formulas for polynomials of degrees up to 4, but there is no formula for polynomials of degree 5 or higher. (The lack of a formula for degree 5 polynomials is one of the celebrated mathematical results of the nineteenth century.) But we don't need a formula, only an algorithm.

There are algorithms for finding zeros of polymonials of arbitrarily high degree. The details are beyond the scope of this class, but you can get a rough idea of how such an algorithm can work. The coefficient with largest absolute value and the polynomial's degree allow you to compute upper and lower bounds on potential zeros. Outside that range, the polynomial is heading toward ∞ or $-\infty$. An algorithm can cut that range up into small pieces and look for an interval where the polynomial changes sign. The polynomial must cross the x-axis somewhere in that interval.

Although we have not proved it here, the real-zero problem is solvable for arbitrary polynomials of a single variable.

9.6.3 Multivariate polynomials

A *multivariate polynomial*, such as $xy - y^2 + 9z$, can have any number of different variables; it is an expression made using variables, integer constants and only operations of addition, subtraction and multiplication.

A single-variable polynomial of degree k can have no more than k different zeros. But a multivariate polynomial can have infinitely many zeros. Look at equation $x - y = 0$. Obviously, any pair of values (x, y) is a zero if $x = y$.

Although the algorithm is very involved, and well beyond the scope of this class, it turns out that the real-zero problem is computable for arbitrary multivariable polynomials.

10 Uncomputable Problems

10.1 Hilbert's Tenth Problem

There is a another problem about multivariate polynomials that is concerned with integer solutions.

Definition 10.1. The *integer-zero problem* takes a multivariate polynomial p with integer coefficients as input and asks whether there exist integer values (members or $\mathcal{Z}$) for the variables that occur in p that make $p = 0$.

In 1900, mathematician David Hilbert posed a list major challenges in mathematics. The tenth problem in the list was to find an algorithm to solve the integer-zero problem or to show that no such algorithm exists. It was not until 1970 that Hilbert's Tenth Problem was solved, in the negative, when Russian mathemetician Yuri Matiyasevich showed that the integer-zero problem is uncomputable.

A proof that Hilbert's Tenth Problem is not computable is far out of reach for us. Matiyasevich relied on work by Martin Davis, Hilary Putnam and Julia Robinson spanning 21 years, and they relied on prior work. But we will be able to prove that some other problems are uncomputable.

10.2 Infinite Loops

Recall that we only say that program p computes function f or language L if p stops on every input. But there are programs that do not stop on every input (that, by definition, do not compute any function or language).

Let's write $p(x)$ to indicate the value that program p returns when it is given input (or parameter) x. Because a program might not always stop, $p(x)$ might not have a value. It is useful to create a special value, $\perp$ (called "bottom"), and say that $p(x) = \perp$ when p runs forever on input x.

You can't know by running p on input x whether it loops forever; it might just take a very, very long time to stop. But, from a mathematical standpoint, p either stops or it doesn't, so either $p(x) = \perp$ or $p(x) \neq \perp$.

When a "value" might be $\perp$, we use relation $\cong$ instead of $=$, where $x \cong y$ is read as "x is equivalent to y."

Definition 10.2. If x and y are strings then $x \cong y \leftrightarrow x = y$. Also, $\perp \cong \perp$. But $x \ncong \perp$ and $\perp \ncong x$ for any string x.

Definition 10.3. $p(x)\downarrow$ (p halts on input x) is equivalent to $p(x) \ncong \perp$. $p(x)\uparrow$ (p does not halt on input x) is equivalent to $p(x) \cong \perp$.

10.3 Interpreters

Your familiarity with computers tells you that, except for resource limitations, any computer can run programs written in any programming language. For example, you can run a Python program on a computer by loading a Python interpreter onto it.

Interpreters are important tools of computability theory. An interpreter allows you to take a program (a string) and run it inside some other program. Running a program via an interpreter must produce the same results as running it directly.

We have allowed programs to be written in any sufficiently strong programming language. Whatever that programming language L is, a self-interpreter is an interpreter for L written in L.

Definition 10.4. A *self-interpreter* (or, more briefly, an interpreter) is a program I having the property that, for every program p and string x, $I(p, x) \cong p(x)$.

Theorem 10.5. There exists a program I that is an interpreter.

Because we know that an interpreter exists, it is acceptable to write $p(x)$ within the body of a program, where p is a string that is either a parameter of the program or that is computed by the program. Running p is just a matter to running a fixed program, the interpreter, that can be built into your own program.

10.4 Problems about Programs

Some decision problems ask questions about programs. An easy one is: Does program p contain a variable called z? But consider the following decision problem, analogous to the acceptance problem for FSMs.

Definition 10.6. The *acceptance problem for programs* is the following decision problem.

> **Input.** Program p and string x.
>
> **Question.** Is $p(x) \cong 1$?

An obvious approach to solving the acceptance problem is to run p on input x and see whether the result is 1. But what if p loops forever? Clearly, that approach does not work.

We have seen that the failure of an obvious approach does not allow us to conclude that no algorithm exists. Concluding that the acceptance problem is not computable needs a rock-solid proof. We will give such a proof in a later section.

10.5 An Uncomputable Decision Problem

Now we identify a decision problem that we can prove is uncomputable.

Definition 10.7. The *Halting Problem* is language

$$HLT = \{(p, x) \mid p(x)\downarrow\}.$$

That is, it is the following decision problem.

> **Input.** Program p and string x.
>
> **Question.** Does p ever stop when it is run on input x?

Theorem 10.8. The Halting Problem is not computable.

Proof.

1. The proof is by contradiction. Start by assuming that the Halting Problem is computable.

Know:	The Halting Problem is computable.
Goal:	**F**.

2. Now we know something that uses term *computable*, and that suggests using a definition. By the definition of a computable decision problem, saying that HLT is computable is equivalent to saying that there exists a program r that stops on all inputs and where, for all y,

$$r(y) \cong 1 \iff y \in \text{HLT},$$

$$r(y) \cong 0 \iff y \notin \text{HLT},$$

But HLT is a set or ordered pairs. It only makes sense to ask if $y \in$ HLT if y is an ordered pair. So lets say that $y = (p, x)$.

Know:	There exists a program r that halts on all inputs so that, for all p and x, $r(p,x) \cong 1 \leftrightarrow (p,x) \in$ HLT and $r(p,x) \cong 0 \leftrightarrow (p,x) \notin$ HLT.
Goal:	**F**.

3. When you know there exists something with a particular property, you ask someone else to give you such a thing. Let's do that, and call the program that was given to us r. The fact that r halts on all inputs is implicit in the two equivalences (1) and (2).

Known variables:	r (a program)
Know (1):	For all p and x, $r(p,x) \cong 1 \leftrightarrow (p,x) \in$ HLT.
Know (2):	For all p and x, $r(p,x) \cong 0 \leftrightarrow (p,x) \notin$ HLT.
Goal:	**F**.

4. By the definition of HLT,

$$(p,x) \in \text{HLT} \;\leftrightarrow\; p(x)\!\downarrow .$$

Known variables:	r (a program)
Know (1):	For all p and x, $r(p,x) \cong 1 \leftrightarrow p(x)\!\downarrow$.
Know (2):	For all p and x, $r(p,x) \cong 0 \leftrightarrow p(x)\!\uparrow$.
Goal:	**F**.

5. So far everything has been boilerplate for a proof by contradition. We have only used definitions. Now comes the inspiration. Every programmer knows how to write an infinite loop. We will allow ourselves to write "loop forever" in a program to indicate an infinite loop. Let's define program s as follows.

```
"{s(z):
  if r(z, z) = 1
    loop forever
  else
    return 1
}"
```

That program looks like it comes out of nowhere, but the discussion after this proof gives motivation for defining it. Program s is written to have two properties.

$$r(z, z) \cong 1 \to s(z)\uparrow .$$

$$r(z, z) \cong 0 \to s(z)\downarrow .$$

Both of those properties should be obvious from the definition of s.

Known variables:	r and s (two programs)
Know (1):	For all p and x, $r(p, x) \cong 1 \to p(x)\downarrow$.
Know (2):	For all p and x, $r(p, x) \cong 0 \to p(x)\uparrow$.
Know (3):	For all z, $r(z, z) \cong 1 \to s(z)\uparrow$.
Know (4):	For all z, $r(z, z) \cong 0 \to s(z)\downarrow$.
Goal:	**F**.

6. Since facts (1) and (2) hold for all p and x, they must hold for $p = s$ and $x = s$. Since facts (3) and (4) hold for all z, they must hold for $z = s$. (Here, we make use of the fact that program s is a string.) Making those substitutions yields the following knowledge.

Known variables:	r and s (two programs)
Know (1):	$r(s,s) \cong 1 \leftrightarrow s(s)\downarrow$.
Know (2):	$r(s,s) \cong 0 \leftrightarrow s(s)\uparrow$.
Know (3):	$r(s,s) \cong 1 \rightarrow s(s)\uparrow$.
Know (4):	$r(s,s) \cong 0 \rightarrow s(s)\downarrow$.
Goal:	**F**.

7. Using known facts (3) and then (2), we get

$$\begin{aligned} r(s,s) \cong 1 &\rightarrow s(s)\uparrow \\ &\rightarrow r(s,s) \cong 0 \end{aligned}$$

If $r(s,s) \cong 1$, that leads to a contradiction. ($r(s,s)$ cannot be both 1 and 0.) So it is not possible for $r(s,s) \cong 1$.

Using known facts (4) and then (1), we get

$$\begin{aligned} r((s,s)) \cong 0 &\rightarrow s(s)\downarrow \\ &\rightarrow r(s,s) \cong 1 \end{aligned}$$

If $r(s,s) \cong 0$, that also leads to a contradiction. So it is not possible for $r(s,s) \cong 0$.

But facts (1) and (2) tell us that $r(s,s)$ *must* be either 0 or 1. (After all, either $s(s)\uparrow$ or $s(s)\downarrow$.) So no matter what, we have reached a contradiction, and have proved **F**.

$\diamondsuit$

What was the motivation for program s in step 5? Notice that, later in the proof, we are only concerned with what s does when its parameter z is s. But, when z is s, the definition of s look as follows.

```
"{s(s):
  if r(s,s) = 1
    loop forever
  else
    return 1
}"
```

(That is not really allowed, since we cannot define a function with its parameter being itself, but let's allow it to understand where the definition of s comes from.)

Now remember that $r(p, x) \cong 1$ if and only if $p(x)\downarrow$ because r was chosen to be a program that solves the halting problem. In the if-statement, s asks r whether s halts on input s. If r says that s halts on input s, then s says, not I don't, and enters an infinite loop. If r says that s does not halt on input s, then s says, yes I do, and s halts and returns 1.

In fact, the proof is quite constructive in the sense that, for every program r that purports to solve the Halting Problem, the proof provides an input (s, s) that r answers incorrectly.

10.6 Diagonalization

The above proof that the Halting Problem is uncomputable uses pairs of strings of the form (s, s). If you think about points in the Cartesian plane, points of the form (x, x) are on the diagonal defined by equation $y = x$. Based on that analogy, the proof that the Halting Problem is uncomputable is called a *proof by diagonalization*.

11 Reductions between Problems

A *reduction* is a way of solving one problem, assuming that you already know how to solve another problem.

11.1 Turing Reductions

Suppose that A and B are two computational problems. They can be decision problems or functional problems. We need to formalize the idea that, if we already know how to solve B, we can solve A. (We need to be careful because the definition needs to work even when A and B are uncomputable problems.)

Definition 11.1. A *Turing reduction from A to B* is a program that computes A and that is able to ask questions about B at no cost.

Definition 11.2. Say that $A \le_t B$ provided there exists a Turing reduction from A to B.

11.1.1 Examples of Turing Reductions

Example 11.3. Suppose that

$$\begin{aligned} A &= \{n \in \mathcal{N} \mid n \text{ is even}\} \\ B &= \{n \in \mathcal{N} \mid n \text{ is odd}\}. \end{aligned}$$

The following program is a Turing reduction from A to B.

```
"{even(x):
   if x ∈ B then
      return 0
   else
      return 1
}"
```

Notice that it asks if $x \in B$. That is where it uses B at no cost. Since there exists a Turing reduction from A to B, we know that $A \leq_t B$.

Example 11.4. Suppose that

$$\begin{aligned} A &= \{p \mid p \text{ is a quadratic polynomial in } x \text{ and } p \text{ has a real zero}\} \\ B &= \{p \mid p \text{ is a polynomial in } x \text{ and } p \text{ has a real zero}\} \end{aligned}$$

Notice that B allows p to have any degree. If you are allowed to ask about zeros of polynomials of any degree, it is easy to ask questions about quadratic polynomials. The following program is a Turing reduction from A to B, establishing that $A \leq_t B$.

```
"{has-zero(p):
   if p ∈ B then
      return 1
   else
      return 0
}"
```

Example 11.5. You can do a Turing reduction between two functions. Define $f : \mathcal{N} \times \mathcal{N} \to \mathcal{N}$ to $g : \mathcal{N} \times \mathcal{N} \to \mathcal{N}$ as follows.

$$\begin{aligned} f(m, n) &= m + n \\ g(m, n) &= m \cdot n \end{aligned}$$

Here is a reduction from g to f. It multiplies by doing repeated additions.

```
"{g(m, n):
   y = 0
   for i = 1, . . . , m
      y = f(y, n)
   return y
}"
```

Example 11.6. The preceding three examples showed how to define a Turing reduction between two *computable* problems. For them, you could just

replace the test $x \in B$ or the use of $f(p, n)$ by a program that tells you whether $x \in B$ or that computes $f(p, n)$. But the following example shows a Turing reduction between two uncomputable problems. Define

$$\begin{aligned} \text{NOTHLT} &= \{(p, x) \mid p(x)\uparrow\} \\ \text{HLT} &= \{(p, x) \mid p(x)\downarrow\} \end{aligned}$$

The following is a Turing reduction from NOTHLT to HLT.

```
"{loops(p, x):
   if (p, x) ∈ HLT then
      return 0
   else
      return 1
}"
```

It is important to recognize that the test $(p, x) \in$ HLT is done without the need for a program that carries out that test. It is done for free, by the definition of a Turing reduction. That is good because there is no program that solves the halting problem.

11.1.2 Properties of Turing Reductions

Suppose that A and B are computational problems. The following theorem should be obvious. Just use the Turing reduction program.

Theorem 11.7. If $A \leq_t B$ and B is computable, then A is computable.

We can turn that around using a tautology that is related to the law of the contrapositive:

$$(p \wedge q) \rightarrow r \leftrightarrow (p \wedge \neg r) \rightarrow \neg q.$$

Corollary 11.8. If $A \leq_t B$ and A is not computable, then B is not computable.

(A corollary is just a theorem whose proof is obvious or trivial, given a previous theorem.) That suggests a way to prove that a problem B is not computable: choose a problem A that you already know is not computable and show that $A \leq_t B$.

11.1.3 An Intuitive Understanding of Turing Reductions

Definition 11.2 defines what $A \le_t B$ means. But it can be helpful to have an intuitive understanding to go along with the definition. Let's look at problems from a viewpoint where computational problems come in only two levels of difficulty: a computable problem is considered easy and an uncomputable problem is considered difficult. Then, according to Theorem 11.7 and Corollary 11.8, $A \le_t B$ indicates that

(a) A is no harder than B, and

(b) B is at least as hard as A.

If you keep that intuition in mind, you will make fewer mistakes. For example, we have seen that, if A is uncomputable and $A \le_t B$, then B is uncomputable too (since it is at least as difficult as uncomputable problem A). What if A is uncomputable and $B \le_t A$? That only tells you that B is no more difficult than an uncomputable problem. So?

11.2 Mapping Reductions

Mapping reductions are a restricted form of reductions that only work for decision problems, but that have some advantages over Turing reductions for decision problems. (One of those advantages is brevity; mapping reductions are often short and simple.) Suppose that A and B are languages (decision problems).

Definition 11.9. A *mapping reduction* from A to B is a *computable* function f such that, for every x,

$$x \in A \leftrightarrow f(x) \in B$$

.

Definition 11.10. Say that $A \le_m B$ provided there exists a mapping reduction from A to B.

11.2.1 Examples of Mapping Reductions

Example 11.11. Suppose that

$$\begin{aligned} A &= \{n \in \mathcal{N} \mid n \text{ is even}\} \\ B &= \{n \in \mathcal{N} \mid n \text{ is odd}\} \end{aligned}$$

Function $f(x) = x+1$ is a mapping reduction from A to B. There is no need to write a program (except to observe that $f(x)$ is computable).

Example 11.12. Suppose that

$$\begin{aligned} A &= \{p \mid p \text{ is a quadratic polynomial in } x \text{ and } p \text{ has a real zero}\} \\ B &= \{p \mid p \text{ is a polynomial in } x \text{ and } p \text{ has a real zero}\} \end{aligned}$$

Then $f(x) = x$ is a mapping reduction from A to B.

Example 11.13. Define

$$\begin{aligned} K &= \{p \mid p(p)\downarrow\} \\ \text{HLT} &= \{(p, x) \mid p(x)\downarrow\} \end{aligned}$$

Then $f(p) = (p, p)$ is a mapping reduction from K to HLT. Notice that

$$\begin{aligned} p \in K &\leftrightarrow p(p)\downarrow \\ &\leftrightarrow (p, p) \in \text{HLT}. \end{aligned}$$

showing that the requirement $p \in K \leftrightarrow f(p) \in \text{HLT}$ of a mapping reduction from K to HLT is met.

11.2.2 Properties of Mapping Reductions

Mapping reductions share some properties with Turing reductions.

Theorem 11.14. If $A \le_m B$ and B is computable then A is computable.

Proof. Suppose that $A \le_m B$. That is, there exists a mapping reduction from A to B. Ask someone else to provide a mapping reduction f from A to B. The following program is a Turing reduction from A to B.

```
"{a(x):
   y = f(x)
   if y ∈ B
      return 1
   else
      return 0
}"
```

Since $x \in A \leftrightarrow f(x) \in B$, it should be clear that $a(x)$ computes A. So $A \leq_t B$. Now simply use Theorem 11.7.

♢

Corollary 11.15. If $A \leq_m B$ and A is not computable then B is not computable.

11.3 Using Turing Reductions to Find Mapping Reductions

Students often find it easier to discover Turing reductions than mapping reductions. One way to discover a mapping reduction is to find a Turing reduction first and to convert that to a mapping reduction. You just need to obey two requirements in the Turing reduction from A to B.

(a) The Turing reduction must only ask one question about whether a string y is in B.

(b) The answer that the Turing reduction returns must be the same as the answer (0 or 1) returned by the test $y \in B$.

If you obey those requirements, then you find that your Turing reduction must have the form

```
"{a(x):
   y = f(x)
   if y ∈ B
      return 1
```

```
    else
      return 0
}"
```

The mapping reduction is f.

We showed earlier that, if A and B are defined by

$$\begin{aligned} A &= \{(p, x) \mid p(x)\uparrow\} \\ B &= \{(p, x) \mid p(x)\downarrow\} \end{aligned}$$

then $A \leq_t B$. It is worth noting that $A \not\leq_m B$. The reason is that any Turing reduction from A to B must negate the answer that it gets to the question about membership in B, and that is not allowed in a mapping reduction.

12 Using Reductions to Show that Problems are Not Computable

Section 11 provides two tools, Turing reductions and mapping reductions, that we can use to demonstrate that a problem is uncomputable. They are generally much easier to apply than diagonalization. Here are the important facts about reductions from Section 11.

Corollary 11.8. If $A \le_t B$ and A is not computable, then B is not computable.

Corollary 11.15. If $A \le_m B$ and A is not computable then B is not computable.

12.1 $p(x)\uparrow$?

Section 11 defines

$$\begin{aligned} \text{NOTHLT} &= \{(p, x) \mid p(x)\uparrow\} \\ \text{HLT} &= \{(p, x) \mid p(x)\downarrow\} \end{aligned}$$

and shows that NOTHLT $\le_t$ HLT. We know from Section 10 that HLT is uncomputable. Relationship NOTHLT $\le_t$ HLT only tells us that NOTHLT is no harder than an uncomputable problem, which tells use nothing about NOTHLT. But it is easy to turn that particular reduction around.

Theorem 12.1. HLT $\le_t$ NOTHLT.

Proof. The following is a Turing reduction from HLT to NOTHLT, establishing that HLT $\le_t$ NOTHLT.

```
"{halts(p, x):
   If (p, x) ∈ NOTHLT then
      return 0
   else
      return 1
}"
```

By Corollary 11.8, since HLT is uncomputable, NOTHLT is also uncomputable.

$\diamondsuit$

12.2 The Acceptance Problem

The acceptance problem for programs is as follows.

$$ACC = \{(p, x) \mid p(x) \cong 1\}.$$

Theorem 12.2. ACC is uncomputable.

Proof. It suffices to show that HLT $\leq_t$ ACC. Here is a Turing reduction from HLT to ACC. It introduces a new wrinkle: it builds a program on the fly.

```
"{halts(p, x):
   r = "{r(z): w = p(z); return 1}"
   if (r, x) ∈ ACC
      return 1
   else
      return 0
}"
```

Clearly

$$\begin{aligned} (r, x) \in ACC &\leftrightarrow r(x) \cong 1 && \text{by the definition of ACC} \\ &\leftrightarrow p(x)\downarrow && \text{by the definition of } r \\ &\leftrightarrow (p, x) \in HLT && \text{by the definition of HLT} \end{aligned}$$

so program halts(p, x) correctly answers the question: is $(p, x) \in$ HLT?

$\diamondsuit$

There is really no need for the full power of a Turing reduction here. Function

$$f(p, x) = (\texttt{"}\{r(z) : w = p(z); \text{return } 1\}\texttt{"}, x)$$

is a mapping reduction from HLT to ACC; f is computable and, as we have just shown,

$$(p, x) \in HLT \leftrightarrow f(p, x) \in ACC.$$

You should begin to recognize the brevity of mapping reductions.

12.3 Does p Terminate on Input 1?

We have seen the trick of creating a program on the fly. With the next reduction, we introduce another trick: make that program ignore its parameter, so that it does the same thing on all strings. To that end, define

$$T_1 = \{r \mid r(1)\downarrow\}.$$

That is, instead of asking whether a give program halts on some given string x, T_1 asks whether the program halts on input 1. That might sound easier than the Halting Problem, but it is not.

Theorem 12.3. T_1 is uncomputable.

Proof. It suffices to show that there is a mapping reduction from HLT to T_1; that is, we show that HLT $\le_m T_1$. The usual way to show that something exists is to produce one, and that is what we do. The following function f is a mapping reduction from HLT to T_1.

$$f(p, x) = \texttt{"}\{r(q) : w = p(x);\ \text{return}\ \ 1\}\texttt{"}.$$

Certainly, f is computable. All it does is write a program (a string) and return that program. f does not run the program that it builds. Notice that program $r(q)$ runs program p on input x, but ignores the result. Also notice that $r(q)$ ignores q; r does the same thing regardless of the parameter that is passed to it.

Let's refer to program "$\{r(q) : w = p(x);\ \text{return } 1\}$" as $r_{p,x}$, acknowledging the fact that p and x are built into r, and you cannot write $r_{p,x}$ until you know what p and x are. Notice that

$$\begin{array}{rcll} (p,x) \in \text{HLT} & \to & p(x)\downarrow & \text{by the definition of HLT} \\ & \to & r_{p,x}(q)\downarrow \text{ for every } q & \text{by the definition of } r_{p,x} \\ & \to & r_{p,x}(1)\downarrow & \\ & \to & r_{p,x} \in T_1 & \text{by the definition of } T_1 \end{array}$$

and

$$\begin{array}{rcll} r_{p,x} \in T_1 & \to & r_{p,x}(1)\downarrow & \text{by the definition of } T_1 \\ & \to & p(x)\downarrow & \text{by the definition of } r_{p,x} \\ & \to & (p,x) \in \text{HLT} & \text{by the definition of HLT} \end{array}$$

Putting those together:

$$(p, x) \in \text{HLT} \;\leftrightarrow\; r_{p,x} \in T_1.$$

Since $f(p, x) = r_{p,x}$, that is exactly the requirement for f to be a mapping reduction from HLT to T_1.

$\diamondsuit$

12.4 Does p Terminate on Input 2?

Define

$$T_2 = \{r \mid r(2)\downarrow\}.$$

It should be obvious how to modify the proof of Theorem 12.3 to show that HLT $\le_m T_2$. But we already know that T_1 is uncomputable, so showing that $T_1 \le_m T_2$ is enough to show that T_2 is uncomputable. Let's do that.

Theorem 12.4. $T_1 \le_m T_2$.

Proof. All we need to do is to transform a question of whether a program a halts on input 1 into an equivalent question of whether another program b_a halts on input 2. That is easy to do: define

$$b_a = \texttt{"}\{b(q):\ \text{return } a(1)\}\texttt{"}.$$

Clearly,

$$a(1)\downarrow \;\leftrightarrow\; b_a(2)\downarrow.$$

That is,

$$a \in T_1 \;\leftrightarrow\; b_a \in T_2.$$

So $f(a) = b_a$ is mapping reduction from T_1 to T_2.

$\diamondsuit$

12.5 The Everything Problem for Programs

Define

$$\text{ALL} = \{p \mid \forall x(p(x)\downarrow)\}.$$

That is, ALL is the following decision problem.

Input. Program p.

Question. Does p halt on every input?

Theorem 12.5. ALL is uncomputable.

Proof. It certainly is not enough to argue that an algorithm to solve ALL would need to try every input. That is nonsense. Suppose stopper is a program that clearly halts on every input.

```
"{stopper(x)
   return 1
}"
```

Do you need to try it on every input to be sure that it stops on every input? Of course not. Consider another program that clearly loops forever on all inputs, such as the following.

```
"{looper(x)
   while(1)
      do nothing
}"
```

You can see from the structure of the program that it loops forever on all inputs. What we need to show is that there is no program R that takes *any* program p as an input and tells you whether p stops on all inputs.

The proof is a mapping reduction from T_1 to ALL. Define

$$\begin{aligned} r_p &= \text{"}\{r(q): \text{ return } p(1)\}\text{"} \\ f(p) &= r_p \end{aligned}$$

Since r_p ignores its parameter q, it should be clear from the definition of r_p that

$$p(1)\downarrow \ \leftrightarrow \ \forall q(r_p(q)\downarrow).$$

That is,

$$p \in T_1 \ \leftrightarrow \ r_p \in \text{ALL}$$

which means that $f(p) = r_p$ is a mapping reduction from T_1 to ALL.

◊

12.6 Complementation and Computability

It is easy to relate the computability of language S and its complement, language $\overline{S}$.

Theorem 12.6. Suppose S is a language over alphabet Σ. If S is a computable then $\overline{S}$ is also computable.

Proof. Suppose that program p computes S. That is, p stops on every input and, for every $x \in \Sigma^*$,

$$p(x) \cong 1 \;\leftrightarrow\; x \in S.$$

The following program computes $\overline{S}$ by flipping answers from 1 to 0 and from 0 to 1.

```
"{Sbar(x):
   if p(x) == 1
      return 0
   else
      return 1
}"
```

In fact, it is obvious that Theorem 12.6 extends to an equivalence.

Theorem 12.7. S is computable if and only if $\overline{S}$ is computable.

12.7 Rice's Theorem

Excluding the proof of Theorem 12.1, you should notice similarities in the above proofs. Excepting only HLT, all of the problems that we looked at are questions about programs, and those questions only depend on what the program does when you run it.

Can we prove a general theorem that takes the similarities of those proofs into account, so that those theorems all become corollaries of the general theorem? Such a theorem would say something like, "It is not computable to determine whether a program has a property that is based solely on what that program does when you run it." We can do something like that, but it is much too vague. The first step we need to make is to find a precise definition of what it means for a set of programs to depend only on what a program does when you run it.

12.7.1 Definitions and Some Obvious Theorems

Definition 12.8. Programs p and q are *equivalent* if $p(x) \cong q(x)$ for every x. That is, the result of $p(x)$ is the same as the result of $q(x)$ for every x. We write $p \approx q$ to mean that p and q are equivalent programs.

Suppose that L is a set of programs over alphabet Σ. Define $\overline{L} = \Sigma^* - L$.

Definition 12.9. L is *nontrivial* if $L \neq \{\}$ and $L \neq \Sigma^*$. That is, neither L nor $\overline{L}$ is empty.

The following theorem is obvious.

Theorem 12.10. L is nontrivial if and only if $\overline{L}$ is nontrivial.

The next definition is critical to what we are trying to do. Read it and make sure that you understand what it says.

Definition 12.11. Suppose L is a set of programs. Say that L *respects equivalence* provided, for every pair of equivalent programs p and q, either p and q are both in L or p and q are both in $\overline{L}$. That is, L must classify any two equivalent programs the same way; they are either both in L or both not in L.

The following is immediate from Definition 12.11.

Theorem 12.12. L respects equivalence if and only if $\overline{L}$ respects equivalence.

Definition 12.13. Define

$$\text{LOOP} = \texttt{"}\{\text{LOOP}(x) : \text{loop forever}\}\texttt{"}$$

to be a program that loops forever on all inputs.

12.7.2 Rice's Theorem

Our goal is to prove a result called Rice's Theorem, which states that every nontrivial set of programs that respects equivalence is uncomputable. We will do that using a lemma and a corollary to the lemma.

Lemma 12.14. If L is a nontrivial set of programs that respects equivalence, and LOOP $\notin L$, then HLT $\leq_m L$. (That is, L is at least as difficult as uncomputable set HLT.)

Proof.

1. Suppose that L is a nontrivial set of programs that respects equivalence and where LOOP $\notin L$.

Known variables:	L
Know (1):	L is a set of programs.
Know (2):	L is nontrivial.
Know (3):	L respects equivalence.
Know (4):	$LOOP \notin L$.
Goal:	HLT $\leq_m L$.

2. Since L is nontrivial, there must be some program that is a member of L. Ask someone else to provide one. Let's call it Y.

Known variables:	L, Y
Know (1):	L is a set of programs.
Know (2):	L is nontrivial.
Know (3):	L respects equivalence.
Know (4):	$LOOP \notin L$.
Know (5):	$Y \in L$.
Goal:	HLT $\leq_m L$.

3. For any given p and x, define $r_{p,x}$ as follows.

"$\{r_{p,x}(z)$:
  $w = p(x)$
  return $Y(z)$
$\}$"

Notice that, for arbitrary p and x,

$$\begin{array}{rcll} (p,x) \in \text{HLT} & \to & p(x)\downarrow & \text{from the definition of HLT} \\ & \to & \forall z(r_{p,x}(z) \cong Y(z)) & \text{from the definition of } r_{p,x} \\ & \to & r_{p,x} \approx Y & \\ & \to & r_{p,x} \in L & \text{since } L \text{ respects equivalence} \end{array}$$

$$\begin{array}{rcll} (p,x) \notin \text{HLT} & \to & p(x)\uparrow & \text{by the definition of HLT} \\ & \to & \forall z(r_{p,x}(z)\uparrow) & \text{by the definition of } r_{p,x} \\ & \to & r_{p,x} \approx \text{LOOP} & \\ & \to & r_{p,x} \notin L & \text{since LOOP} \notin L \text{ and } L \text{ respects equivalence} \end{array}$$

Known variables:	L, Y, $r_{p,x}$
Know (1):	L is a set of programs.
Know (2):	L is nontrivial.
Know (3):	L respects equivalence.
Know (4):	$LOOP \notin L$.
Know (5):	$Y \in L$.
Know (6):	$\forall p \forall x((p,x) \in \text{HLT} \to r_{p,x} \in L)$
Know (7):	$\forall p \forall x((p,x) \notin \text{HLT} \to r_{p,x} \notin L)$
Goal:	$\text{HLT} \leq_m L$.

4. Our mapping reduction from function HLT to L is:

$$f(p,x) = r_{p,x}.$$

Clearly, f is computable, since it only needs to write down program $r_{p,x}$. Putting facts (6) and (7) together,

$$(p,x) \in \text{HLT} \leftrightarrow r_{p,x} \in L.$$

So f is a mapping reduction from HLT to L.

$\diamondsuit$

Corollary 12.15. If L is a nontrivial set of programs that respects equivalence, where LOOP $\notin L$, then L is not computable.

Proof. That follows immediately from Lemma 12.14, corollary 11.15 and the fact that HLT is uncomputable.

$\diamondsuit$

Theorem 12.16. (Rice's Theorem) If L is a nontrivial set of programs that respects equivalence, then L is not computable.

Proof. There are two cases: either LOOP $\notin L$ or LOOP $\in L$.

If LOOP $\notin L$, then Theorem 12.16 follows immediately from Corollary 12.15.

So consider the case where LOOP $\in L$. Then LOOP $\notin \overline{L}$. By Theorems 12.10 and 12.12, $\overline{L}$ is nontrivial and $\overline{L}$ respects equivalence. So $\overline{L}$ meets the requirements of Corollary 12.15. We conclude that, in this case, $\overline{L}$ is uncomputable. By Theorem 12.7, L is also uncomputable.

$\diamondsuit$

12.8 Examples Using Rice's theorem

12.8.1 Example: T_1 is Uncomputable

Recall that we defined

$$T_1 = \{r \mid r(1)\downarrow\}.$$

Let's reprove that T_1 is uncomputable using Rice's Theorem.

Theorem 12.17. T_1 is uncomputable.

Proof. Since some programs halt on input 1 and some don't, T_1 is nontrivial. Suppose that p and q are two equivalent programs. Then

$$\begin{array}{rcll} p \in T_1 & \leftrightarrow & p(1)\downarrow & \text{by the definition of } T_1 \\ & \leftrightarrow & q(1)\downarrow & \text{since } p \approx q \\ & \leftrightarrow & q \in T_1 & \text{by the definition of } T_1 \end{array}$$

So T_1 respects equivalence. By Rice's Theorem, T_1 is uncomputable.

$\diamondsuit$

12.8.2 Example: Is $L(p)$ finite?

Define

$$\text{FINITE} = \{p \mid L(p) \text{ is a finite set}\}.$$

FINITE is the following decision problem.

> **Input.** A program p.
>
> **Question.** Is $L(p)$ finite? That is, is $\{x \mid p(x) \cong 1\}$ a finite set?

Notice that FINITE is not a finite set! It is a set of programs. For every computable set S, there are infinitely many programs that solve S. (You can make infinitely many variations on a program without changing the set that it decides.) So there are infinitely many programs p where $L(p) = \{\}$, and all of those are members of FINITE.

Theorem 12.18. FINITE is uncomputable.

Proof. FINITE is nontrivial. Some programs answer 1 on only finitely many inputs, and some answer 1 on infinitely many inputs.

Suppose that p and q are two equivalent programs. Then

$$\begin{aligned} p \in \text{FINITE} &\leftrightarrow L(p) \text{ is a finite set} \\ &\leftrightarrow L(q) \text{ is a finite set} \quad \text{since } p \approx q \\ &\leftrightarrow q \in FINITE \end{aligned}$$

So FINITE respects equivalence. By Rice's Theorem, FINITE is uncomputable.

♢

12.8.3 Example: Is $L(p) = \{\}$?

Define

$$\text{EMPTY} = \{p \mid L(p) = \{\}\}.$$

EMPTY is not an empty set! It is the following decision problem.

> **Input.** A program p.
>
> **Question.** Is it the case that $L(p) = \{\}$? That is, are there no inputs x on which p stops and answers 1?

Theorem 12.19. EMPTY is uncomputable.

Proof. EMPTY is clearly nontrivial. It also respects equivalence.

$$\begin{aligned} p \in \text{EMPTY} &\leftrightarrow L(p) = \{\} \\ &\leftrightarrow L(q) = \{\} \qquad \text{since } p \approx q \\ &\leftrightarrow q \in EMPTY \end{aligned}$$

By Rice's Theorem, EMPTY is uncomputable.

$\diamondsuit$

12.9 Are p and q equivalent?

Define

$$\text{EQUIV} = \{(p, q) \mid p \approx q\}.$$

Rice's Theorem has nothing to say about EQUIV because EQUIV is not a set of programs. It is a set of ordered pairs of programs. Nevertheless, we can show that EQUIV is uncomputable.

Theorem 12.20. EQUIV is uncomputable.

Proof. Define

$$\text{NEVERHALT} = \{p \mid \forall x(p(x)\uparrow)\}.$$

NEVERHALT is a nontrivial set of programs that respects equivalence. Rice's theorem tells us that NEVERHALT is uncomputable. An equivalent definition is:

$$\text{NEVERHALT} = \{p \mid p \approx \text{LOOP}\}.$$

Function f defined by

$$f(p) = (p, \text{LOOP})$$

is a mapping reduction from NEVERHALT to EQUIV, since

$$\begin{aligned} p \in \text{NEVERHALT} &\leftrightarrow p \approx \text{LOOP} \\ &\leftrightarrow (p, \text{LOOP}) \in \text{EQUIV} \end{aligned}$$

12.10 K

Define

$$K = \{p \mid p(p)\downarrow\}.$$

K is a set of programs, but it does not respect equivalence. Let's try to show that K respects equivalence to see where the proof breaks down.

$$\begin{aligned} p \in K &\leftrightarrow p(p)\downarrow \\ &\leftrightarrow q(p)\downarrow \quad \text{since } p \approx q \end{aligned}$$

But what q does on input p is irrelevant to determining whether $q \in K$. All that matters is what q does on input q.

Nevertheless, we can show:

Theorem 12.21. K is uncomputable.

Proof. Rice's theorem is not a help here. But it suffices to show that HLT $\leq_m$ K. For arbitrary p and x, define $r_{p,x}$ as follows.

"{$r_{p,x}(z)$:
$w = p(x)$
return 1
}"

Notice that $r_{p,x}$ ignores its parameter, z. It just runs $p(x)$. It is evident that

$$\begin{aligned} (p,x) \in HLT &\rightarrow p(x)\downarrow \\ &\rightarrow \forall z (r_{p,x}(z)\downarrow) \\ &\rightarrow r_{p,x}(r_{p,x})\downarrow) \\ &\rightarrow r_{p,x} \in K \end{aligned}$$

and

$$\begin{aligned} (p,x) \notin HLT &\rightarrow p(x)\uparrow \\ &\rightarrow \forall z (r_{p,x}(z)\uparrow) \\ &\rightarrow r_{p,x}(r_{p,x})\uparrow) \\ &\rightarrow r_{p,x} \notin K \end{aligned}$$

which tells us that

$$f(p,x) = r_{p,x}$$

is a mapping reduction from HLT to K.

12.11 Concrete examples

Without concrete examples, it can be easy to believe that our theorems about problems being uncomputable are only of abstract, mathematical significance, and have no bearing on the real world. So let's look at some concrete examples to see that the real world is not immune to mathematical theorems.

12.11.1 The 3n+1 problem

The $3n + 1$ problem concerns an infinite collection of sequences of integers. Select a positive integer n to start a sequence. Follow it by $n/2$ if n is even and by $3n + 1$ if n is odd. Stop the sequence when it reaches 1. The $3n + 1$ sequence starting with 9 is (9, 28, 14, 7, 22, 11, 34, 17, 52, 26, 13, 40, 20, 10, 5, 16, 8, 4, 2, 1).

It is not obvious that the $3n + 1$ sequence stops for all starting values. It is conceivable that it gets into a cycle. It is also conceivable that, for some starting values, the numbers in the $3n + 1$ sequence keep getting larger and larger, without bound. In fact, nobody knows whether every $3n+1$ sequence is finitely long. But we can always make a conjecture.

Conjecture 12.22. The $3n+1$ sequence is finitely long for every start value.

Look at the following program.

```
"{test(n):
   i = n
   while i > 1
      if i is even
         i = i/2
      else
         i = 3i + 1
}"
```

Can you tell whether test is in ALL? (That is, does test halt on all inputs x?) If test is in ALL, then Conjecture 12.22 is true. If not, then Conjecture

12.22 is false. If you can write a computer program that solves ALL, then that program tells you whether the above conjecture is true.

But that seems unreasonable; a computer should not be able to resolve a deep conjecture like that. The fact that ALL is uncomputable keeps you from solving a deep conjecture by running a computer program that seems to have nothing to do with the conjecture.

12.11.2 Does program p test whether a number is prime?

Now suppose that you are serving as a grader for a computer programming course. One of the assignments for that course asks students to write a program that reads an integer $n > 1$ and tells whether n is prime. As grader, you are tasked with determining whether each submission is correct, with the sole criterion for correctness being that the program correctly determines whether n is prime for every integer n. (In the programming language being used, integers can be arbitrarily large, so you can't try the program on a finite range of integers to decide whether it works.)

To make sure that you are ready, you write your own program p to tell if a number is prime. Now, given a student submission q, the problem is to determine whether $q \approx p$. But that is uncomputable! Could that possibly be a problem? Suppose that a particularly devious student submits the following program.

```
"{q(n)
   i = n
   while i > 1
      if i is even
         i = i/2
      else
         i = 3i + 1
   i = 2
   while i < n
      if n mod i == 0
         return 0
      i = i + 1
   return 1
}"
```

You notice that, if Conjecture 12.22 is true, the submitted program q is correct. But if Conjecture 12.22 is false, then there are values n on which q loops forever, meaning that q is incorrect. In order to grade q according to the grading criterion, you must determine whether Conjecture 12.22 is true!

12.11.3 Goldbach's conjecture

The following conjecture is due to Goldbach.

Goldbach's Conjecture 12.23 Every even integer that is greater than 2 is the sum of two prime integers.

For example, $4 = 2 + 2$, $6 = 3 + 3$, $8 = 3 + 5$, $10 = 5 + 5$, etc. Nobody knows whether Goldbach's conjecture is true, and it appears to be a very difficult nut to crack. But we can write the following program, which contains an infinite loop that checks, for each even number n, whether there are two prime numbers whose sum is n. If it finds an even number n that is not the sum of two prime numbers, it stops. Otherwise, it loops forever.

```
"{goldbach()
   n = 4
   while(1)
      i = 2
      found = 0
      while found == 0 and i < n
         if i is prime and n − i is prime
            found = 1
         i = i + 1
      if found == 0
         return 0
      n = n + 2
}"
```

To answer Goldbach's conjecture, all you need to do is ask whether program goldbach ever stops. You can ask whether it is in ALL or in T_1 or in a variety of languages because goldbach ignores its input.

Goldbach's conjecture is another deep conjecture that could be resolved by running a computer program if ALL or T_1 is computable. The fact that no such computer program exists should come as no surprise.

12.11.4 Compilers

Compilers for programming languages offer warnings when you do something suspicious (some more than others). One warning that would be nice would be whether the program can ever loop forever. Yet, no compilers offer such warnings. Can you say why not?

13 Partially Computable Sets and m-Complete Problems

13.1 Partially Computable Sets

For a program p, we have defined

$$L(p) = \{x \mid p(x) \cong 1\}$$

We also added a stipulation that p can only be said to decide $L(p)$ if p halts on all inputs. What if we remove the stipulation? Does the definition of $L(p)$ make any sense then?

Let's say that a language X is *partially computable* if there exists a program p so that $X = L(p)$, without the stipulation that p must halt on all inputs. Is there a difference between the notion of a computable language and a partially computable language?

It is easy to see that the class of partially computable languages is not the same as the class of computable language. We know that HLT is not computable.

Theorem 13.1. HLT is partially computable.

Proof. Here is a program halt(x) where $L(\text{halt}) = \text{HLT}$.

"{halt(p, x):
  Run $p(x)$
  return 1
}"

You can check that HLT is, indeed, the set of all ordered pairs (p, x) where $\text{halt}(p, x) \cong 1$. What is going on here?

The trick lies in the fact that halts(p, x) answers "no" by looping forever. That is unpleasant because you can't tell the difference between a program that is looping forever and a program that is just taking a very long time to stop.

Notice that every computable language is also partially computable. That is easy. The requirement

$$L(p) = \{x \mid p(x) \cong 1\}$$

is the same as the requirement for a language to be computable. We have just omitted the extra requirement that p must stop on every input.

13.2 M-Complete Problems

BIG IDEA: We can identify hardest problems in particular classes.

What is the point of defining partially computable languages? Well, for one thing, they give us a third level of difficulty. A language can be computable; partially computable (but not computable); and not even partially computable. For example, ALL is known not to be partially computable.

For our purposes, the class of partially computable languages gives us an opportunity to identify languages are the most difficult partially computable problems.

Suppose that you want to show that person t is a tallest person in the room. (It is possible for two people to be equally tall, so there might be more than one tallest person). First, you would obviously need for person t to be in the room. Next, you would need to compare person t's height with the height of every other person in the room and find that t is at least as tall as every person in the room.

We can use an analogous idea to identify most difficult partially computable problems.

Definition 13.2. Say that language T is *m-complete* if both of the following are true.

1. T is partially computable.

2. $X \leq_m T$ for every partially computable language X.

If you think of relation $\leq_m$ as meaning "no more difficult than,", then you can see that an m-complete language is one of the hardest partially computable languages.

Do m-complete languages exist? Yes, and HLT is one of them.

Theorem 13.3. HLT is m-complete.

Proof. We have seen that HLT is partially computable, so condition (1) is met. Now we must show that $X \le_m$ HLT for every partially computable language X.

Suppose that X is partially computable. Select a program r that partially computes X. We can modify r so that it never answers "no" by making it go into an infinite loop instead of answering no. Then we find that

$$X = \{x \mid r(x)\downarrow\}$$

since, for the modified program r, halting is equivalent to answering "yes."

Keep in mind the goal here: show that $X \le_m$ HLT. An m-reduction from X to HLT is

$$f(x) = (r, x).$$

It is easy to check that $f(x)$ has both of the properties needed of an m-reduction from X to HLT. Clearly, f is computable, since it just writes the fixed program r along with x. Also, for every x,

$$x \in X \ \leftrightarrow\ f(x) \in \text{HLT}.$$

♢

It should not be surprising that HLT is one of the most difficult partially computable languages. If you could solve HLT then you could solve every partially computable language; just use your solution to HLT to tell whether or not a program is looping forever, making $\perp$ just as good an answer as 'no'.

A most difficult partially computable language ought not to be computable. Take HLT as an example.

Theorem 13.4. If L is m-complete then L is uncomputable.

Proof. HLT is partially computable. By the definition of an m-complete language, HLT $\le_m L$. By Corollary 11.15, L is not computable. (Intuitively: L is at least as hard as uncomputable problem HLT.)

♢

Later, we will see another context where we want to find problems that are among the most difficult problems in a particular class.

13.3 Co-Partially Computable Sets

There is an asymmetry in the definition of partially computable sets. A language L is partially computable if it is solvable by a program that stops on inputs that are in L and loops forever on inputs that are not in L. What if we define an analogous class of problems that are solvable by programs that stop when the answer is no and loop forever when the answer is yes?

Definition 13.5. Language L is *co-partially computable* if there exists a program p so that stops when x is not in L and that loops forever when x is in L.

Is that the same class as the partially computable languages? No, because of the asymmetry! Look at $\overline{\text{HLT}}$. You already know that there is a program p that stops on an input that is in HLT and loops forever on inputs that are not in HLT. The same program p halts when $x \notin \overline{\text{HLT}}$, and shows that $\overline{HLT}$ is co-partially computable.

The following theorem is enough to show that the partially computable languages cannot be the same as the co-partially computable language.

Theorem 13.6. For every language L, L is computable if and only if L is both partially computable and co-partially computable.

Proof. First, suppose that L is both partially computable and co-partially computable. Get a program p that stops on inputs that are in L and another program q that stops on inputs that are not in L. To decide whether $x \in L$, run $p(x)$ and $q(x)$ concurrently. Eventually, one of them must stop. If the program that stops is p, say that x is in L. If the program that stops is q, say that x is not in L.

Next, suppose that L is computable. It is easy to show that L is both partially computable and co-partially computable. For the former result, just take a program that decides p and make it loop forever when it is about to stop and answer 'no'. For the latter result, make p loop forever when it is about to answer 'yes'.

$\diamondsuit$

Later, we will see another class that shares the asymmetry in its definition as the class of partially computable sets and has a notion of most difficult problems. Whether or not an analog of Theorem 13.6 holds is critical to whether there exist public-key cryptosystems.

14 Computational Complexity and Polynomial Time

A program is only said to compute, or decide, a problem if it eventually stops on every input. For computability, it does not matter how long the program takes to produce an answer.

But, from a practical standpoint, time matters. The data switches that form the backbone of the internet process a data packet roughly every 10-20 nanoseconds. They can only afford to run extremely fast algorithms. For most everyday purposes, an algorithm that gets its answer in a few seconds is fast enough, and for some problems, a program that takes a few minutes or hours is fast enough. But a program that takes years is usually not acceptable.

With this section we start to look at what can be computed efficiently. To do that, we need to choose a reasonable definition of an "efficient" algorithm, and to study which problems can be solved efficiently under that definition.

A definition of efficiency cannot be based on any fixed amount of time. The larger the input is, the longer we expect a program to take, so a reasonable notion of efficiency should be concerned with the time that a program takes as a function of the length of the input. Also, a faster computer will produce an answer faster even for the same algorithm, and we need a way to get processor speed out the way.

14.1 The Class P

A program performs a sequence of instructions, and the time that it uses is just a count of the number of instructions that it performs before it stops. In this view, time has no units; it is a pure number. We assume that it takes at least one instruction to look at a symbol in the input and at least one instruction to write one symbol in the output.

Definition 14.1. *Time(p,x)* is the number of instructions that program p takes when it is run on input x. If $p(x)\uparrow$, then $\text{Time}(p,x) = \infty$.

Definition 14.2. Let $f : \mathcal{N} \to \mathcal{N}$. A program p runs in *time* $O(f(n))$ if there exists constants a and c so that, for all $n > a$ and all strings x of length

n, Time$(p, x) \le c \cdot f(n)$.

Definition 14.3. Program p runs in *polynomial time* if there exists a positive integer k so that p runs in time $O(n^k)$. When p runs in polynomial time, we say that p is a *polynomial-time algorithm*.

Definition 14.4. *P* is the class of all *decision problems* that have polynomial-time algorithms.

Notice that P is a set of problems, not a set of programs. It makes no sense to say that a program is in P.

14.2 Examples of Problems that Are In P

14.2.1 Example: is x a Palindrome?

A *palindrome* is a string such as "*aabaa*" that is the same forwards and backwards. The *Palindrome Problem* is the following decision problem.

Input. String x.

Question. Is x a palindrome?

The Palindrome Problem is in P. You should be able to figure out an algorithm that solves the Palindrome Problem in time $O(n)$.

14.2.2 Example: Does DFA M Accept x?

We have seen that it is computable to determine if a given DFA M accepts a given string x. It should also be clear that an algorithm can do that in time that is proportional to the product of the length of the description of M and the length of x. (You should be able to do much better than that, but it is fast enough for our purposes.) If the input has length n, that is surely $O(n^2)$ time.

14.2.3 Example: Is $x \cdot y = z$?

How might you solve the following decision problem?

Input. Three positive integers x, y and z.

Question. Is $x \cdot y = z$?

The obvious thing to do is to multiply x and y and check whether the answer is z. It is important to notice that the time needed to do that is not a constant, since x, y and z can be very large. When a number occurs in the input, its length is the number of digits needed to write it down. For example, 490 has length 3. The algorithm that you learned in elementary school multiplies an i-digit number by a j-digit number in time $O(ij)$. The length of the input is the total number of symbols that it contains. Clearly, if x has length i and y has length j and the total length of the input if n, then $i < n$ and $j < n$, and it is possible to check an integer product in time $O(n^2)$. That is polynomial time.

14.2.4 Example: Is x a Prime Number?

The *Primality Problem* is the following decision problem.

Input. A positive integer x.

Question. Is x prime?

Here is an algorithm that solves the Primality Problem.

```
"{prime(x):
  i = 2
  while i < x
    if n mod i == 0
      return 0
    i = i + 1
  return 1
}"
```

How much time does that algorithm take? It goes around the loop $x - 2$ times. If x is n digits long, then x is in the rough vicinity of 10^n. (Assuming no leading 0s, $10^{n-1} \le x < 10^n$.) The division algorithm that you learned in elementary school divides an m-digit number by an n-digit number in time $O(mn)$. Puting that all together, we find that our algorithm for testing

whether an integer is prime takes time $O(n^2 10^n)$. But function $f(n) = 10^n$ grows faster than any polynomial. That is, for every k,

$$\lim_{n\to\infty} \frac{10^n}{n^k} = \infty.$$

Our primality-testing algorithm is not a polynomial-time algorithm.

What does that tell us about whether the Primality Problem is in P? Absolutely nothing! You can write a bad algorithm for any computable problem. The issue is not whether there is a bad algorithm to solve the Primality Problem, but whether there is a polynomial-time algorithm for it.

As it turns out, the primality problem is in P. It was long conjectured to be in P, and was shown to be in P in 2003.

14.3 The Validity Problem for Propositional Logic

Chapter 1 defines the notion of a valid propositional formula. The *Validity Problem for Propositional Logic* (or simply the Validity Problem) is the following decision problem.

Input. A propositional formula ϕ.

Question. Is ϕ valid?

You already know an algorithm that solves that problem: truth tables. Suppose that ϕ has v variables. Then a truth table for ϕ has 2^v rows, and determining validity takes time at least 2^v.

Determining whether an algorithm runs in polynomial time requires determining the algorithm's running time in terms of the length n of the input. The number of variables v is surely shorter than the total length of ϕ, but how close to n can v be? As long as we allow long variable names (such as x_1, x_2, ...), it is easy to write an interesting propositional formula of length n with at least $\sqrt{n}$ variables. A little calculus shows that

$$\lim_{n\to\infty} \frac{2^{\sqrt{n}}}{n^k} = \infty$$

for every k, so the truth table algorithm does not run in polynomial time.

What does that have to say about whether the Validity Problem is in P? Absolutely nothing! Nobody knows a polynomial-time algorithm for the

Validity Problem, but that lack of knowledge also is not convincing evidence that the Validity Problem is not in P.

Beginning with the next section, we begin to address the question of whether the Validity Problem is in P.

15 Nondeterminism and NP

15.1 A Larger Class than P

In Section 13, we saw that it can be useful to have a class of problems that is a little larger than the class that you are interested in. For example, that allows you to identify problems that are among the most difficult problems in the larger class. If class C is a subset of class D and X is one of the most difficult problems in the larger class D, you would expect X not to be in the smaller class C.

In this section, we introduce a class NP that (we hope) is larger than P.

15.2 Mersenne's Conjecture

In 1644, Marin Mersenne made what came to be known as Mersenne's conjecture: $2^n - 1$ is prime for $n = 2$, 3, 5, 7, 13, 17, 19, 31, 67, 127 and 257. and is composite for all other positive integers $n \leq 257$.

Mersenne's conjecture stood until 1903 when Frank Cole made a presentation that put it to rest. The presentation was quite short. By starting with 2, successively doubling and finally subtracting 1, Cole showed that

$$2^{67} - 1 = 147,573,952,589,676,412,927.$$

Then he wrote down two numbers and multiplied them together.

$$\begin{array}{lr} & 761{,}838{,}257{,}287 \\ \times & 193{,}707{,}721 \\ \hline & 147{,}573{,}952{,}589{,}676{,}412{,}927 \end{array}$$

That was all it took to convince Cole's audience that Mersenne's conjecture was mistaken. (Other mistakes in it were discovered later.)

But where did Cole get the factors of $2^{67} - 1$? He said that it took "three years of sundays" to find them.

In idealized form, our system of justice is supposed to work as follows. First, the police gather evidence. Then, the prosecutor presents the case to a jury.

Finally, the jury rules on whether the evidence is convincing. If the jury does not find the evidence convincing, the jurors are not required to go out and find new evidence. The case is over.

Frank Cole played the role of police and prosecutor and his audience played the role of jury. It can take much less time to present a case than it does to find the evidence.

15.3 Evidence Checkers

We can break down testing whether string x is in language A into two parts: finding evidence and checking the evidence.

Definition 15.1. A *polynomial-time evidence checker* for language A is a program $\text{check}(e, x)$ where there exists a positive integer k so that

1. $\text{check}(e, x)$ runs in polynomal time (in the length of ordered pair (e, x)).
2. If $x \in A$ then there is a string e (the evidence) where $|e| \leq |x|^k$ and $\text{check}(e, x) = 1$. That is, members of A have short, easy to check evidence that they are members of A. (The jury correctly recognizes convincing evidence.)
3. If $x \notin A$ then there does not exist any string e so that $\text{check}(e, x) = 1$. That is, the evidence checker cannot be fooled into believing that $x \in A$ when in fact $x \notin A$. (The jury does not convict on bad evidence.)

There is an important asymmetry in evidence checkers. If $x \in A$, then there must be checkable evidence that $x \in A$. But if $x \notin A$, no evidence is required showing that $x \notin A$. All that is required is a lack of evidence that $x \in A$.

15.4 Definition of NP

Definition 15.2. *NP* is the class of all decision problems that have polynomial-time evidence checkers.

For example, an integer x is *composite* if $x > 1$ and x is not prime. The smallest composite number is 4 $(= 2 \cdot 2)$. Define

$$\text{COMPOSITE} = \{x \in \mathcal{N} \mid x \text{ is composite}\}.$$

It is easy to see that COMPOSITE is in NP. (Frank Cole showed how.) The following is a polynomial-time evidence checker for COMPOSITE where the evidence e should be a factor of x and the checker verifies that.

```
"{composite(e, x):
  If 1 < e < x and x mod e == 0
    return 1
  else
    return 0
}"
```

A simpler way to present an evidence checker is to list (1) the input, (2) the evidence and (3) the conditions that needs to be satisfied for the evidence to be convincing.

Evidence checker for COMPOSITE	
Input	Positive integer x
Evidence	Positive integer e
Requirement	$1 < e < x$ and $x \bmod e = 0$

15.5 Examples of Problems In NP

15.5.1 Is a given propositional formula satisfiable?

Definition 15.3. A propositional formula ϕ is *satisfiable* if there exists a truth-value assignment for the variables in ϕ that makes ϕ true.

Definition 15.4. The *Satisfiability Problem for Propositional Logic* (SATPL) is the following decision problem.

Input. A propositional formula ϕ.
Question. Is ϕ satisfiable?

Theorem 15.5. SATPL is in NP.

Proof. All we need is a polynomial-time evidence checker for SATPL. If you think about a truth-table for ϕ, you only need to look at one row to determine that ϕ is satisfiable.

Evidence checker for SATPL	
Input.	Propositional formula ϕ
Evidence.	Truth value assignment a
Requirement.	$(a \dashv \phi)$ is true.

$\diamondsuit$

15.5.2 Does a Graph Have a Small Vertex Cover?

Definition 15.6. Suppose that $G = (V, E)$ is a simple graph. A *vertex cover* of G is a subset $C \subseteq V$ such that, for every edge $\{u, v\} \in E$, $C \cap \{u, v\} \neq \{\}$. That is, every edge must be incident on at least one member of the vertex cover C.

Definition 15.7. The *Vertex Cover Problem* (VCP) is the following decision problem.

Input. A simple graph G and a positive integer k.
Question. Does there exist a vertex cover C of G where $|C| \leq k$?

Theorem 15.8. VCP $\in$ NP.

Proof. Decision problems in NP are often stated as a question about whether something exists. For example, VCP asks whether a vertex cover of a limited size exists. To find a polynomial-time evidence checker, use the thing whose existence is questioned as the evidence. The following is an evidence checker for VCP.

Evidence checker for VCP	
Input	Simple graph $G = (V, E)$ and positive integer k
Evidence	Set $C \subseteq V$
Requirement	$\lvert C \rvert \leq k$ and C is a vertex cover of G.

$\diamondsuit$

15.5.3 Can a List of Integers Be Partitioned Equally?

Definition 15.9. A list of positive integers x_1, x_2, ..., x_n is *equally partitionable* if there exists an index set $I \subseteq \{1, 2, \ldots n\}$ such that

$$\sum_{i \in I} x_i = \sum_{i \notin I} x_i.$$

For example, suppose the list of integers is $x_1 = 14$, $x_2 = 10$, $x_3 = 5$, $x_4 = 7$, $x_5 = 2$, $x_6 = 4$, $x_7 = 6$. Index set $\{1, 6, 7\}$ equally partitions that list since $x_1 + x_6 + x_7 = 14 + 4 + 6 = 24$ and $x_2 + x_3 + x_4 + x_5 = 10 + 5 + 7 + 2 = 24$.

Definition 15.10. The Partition Problem (PP) is the following decision problem.

Input. A list x_1, x_2, ..., x_n of positive integers.
Question. Is x_1, x_2, ..., x_n equally partitionable?

Theorem 15.11. PP $\in$ NP.

Proof. List x_1, x_2, ..., x_n is equally partitionable if *there exists* an index set I so that

$$\sum_{i \in I} x_i = \sum_{i \notin I} x_i.$$

That suggests using I as the evidence.

Evidence checker for PP	
Input	List x_1, x_2, ..., x_n of positive integers
Evidence	Index set $I \subseteq \{1, \ldots n\}$
Requirement	$\sum_{i \in I} x_i = \sum_{i \notin I} x_i$

◊

15.6 Every Problem In NP Is Computable

Theorem 15.12. If $A \in$ NP then A is computable.

Proof. Suppose that $A \in \text{NP}$. Let $c(e, x)$ be a polynomial-time evidence checker for A. By the definition of a polynomial-time evidence checker, there is an integer k so that $x \in A$ if and only if there exists a string e with $|e| \leq |x|^k$ and $c(e, x) = 1$.

An algorithm can decide whether $x \in A$ by computing $c(e, x)$ for every string e where $|e| \leq x^k$, answering yes if any of those yields 1.

♢

15.7 Every Problem In P Is Also In NP

Theorem 15.13. Every language that is in P is also in NP. That is, $\text{P} \subseteq \text{NP}$.

Proof. Suppose that A is a language in P. By definition, that means there is a polynomial-time algorithm $\text{inA}(x)$ where $\text{inA}(x) = 1 \leftrightarrow x \in A$. The following is a polynomial-time evidence checker for A. It does not need the evidence, so it ignores the evidence.

Evidence checker for A	
Input	x
Evidence	any string e
Requirement	$\text{inA}(x) = 1$

♢

15.8 The P = NP Question

Think of a problem in NP as a kind of puzzle. Solving a puzzle requires finding a solution, which amounts to evidence that the puzzle has a solution. Often, the solution for a puzzle in the newspaper or a book is provided, and peeking at the solution is much less time consuming (though less satisfying) than finding the solution yourself.

Theorem 15.13 tells us that $\text{P} \subseteq \text{NP}$. But is $\text{NP} \subseteq \text{P}$? If $\text{NP} \subseteq \text{P}$, then, at least up to a polynomial, peeking at the solution is not helpful; you can just find the solution yourself.

If NP $\subseteq$ P then P = NP. Surprisingly, nobody knows whether P = NP. It is widely *conjectured* that P $\neq$ NP, but we have already seen that a conjecture can stand for over 200 years only to be overturned.

16 NP-Completeness

16.1 "Easy" and "Hard" Problems

In Section 11, we only considered two levels of difficulty of problems: computable problems and uncomputable problems. Starting with Section 14, we have begun by considering only just two levels of difficulty, where we think of a problem as "easy" if it is in P and as "hard" if it is not in P. Let's refer to the class of decision problems that are not in P (the "hard" problems) as $\overline{\text{P}}$.

The previous section introduced a third level of difficulty, NP. A problem that is in P is also in NP and every problem that is in NP is computable.

A common misconception is that NP = $\overline{\text{P}}$. That is not the case at all. Can you think of a problem that is in $\overline{\text{P}}$ but not in NP? How about the Halting Problem (HLT)? Since every problem in NP is computable (Theorem 15.12), HLT is not in NP. HLT is also not in P, so it is in $\overline{\text{P}}$.

Our ultimate goal is to find problems that are in NP but not in P. That is, they are not in P, but are only slightly outside of P since they are in NP. This section identifies "hardest" problems in NP, which are the best candidates for languages that are in NP but not in P. In overview:

1. We define polynomial-time mapping reductions and relation $A \leq_p B$ where, if $A \leq_p B$ and $B \in \text{P}$ then $A \in \text{P}$. Intuitively, you can think of $A \leq_p B$ as saying that B is at least as hard as A (in our new two levels of difficulty P and $\overline{\text{P}}$).

2. We define a problem to be NP-complete if it is among the hardest problems in NP. That is, it must be in NP and it must be at least as hard as every other problem in NP.

3. Section 17 identifies some NP-complete problems. In this section, we look at the consequences of a problem being NP-complete.

16.2 Polynomial-Time Mapping Reductions

BIG IDEA: Polynomial-time mapping reductions are a special case of mapping reductions that give a closer relationship between problems than the mapping reductions that we saw earlier.

Definition 16.1. Suppose that A and B are languages (decision problems). A *polynomial-time mapping reduction from A to B* is a function f where

(a) f is computable in polynomial time.

(b) For every string x, $x \in A \leftrightarrow f(x) \in B$.

Definition 16.2. We say that $A \leq_p B$ if there exists a polynomial-time mapping reduction from A to B.

The only difference between a polynomial-time mapping reduction and the mapping reductions defined in Section 11 is the requirement that f must not merely be computable, but must be computable in polynomial time. It should come as no surprise that polynomial-time mapping reductions have properties that are similar to unrestricted mapping reductions, but with respect to P and $\overline{\text{P}}$ rather than with respect to computable and uncomputable problems.

Theorem 16.3. If $A \leq_p B$ and $B \in \text{P}$ then $A \in \text{P}$.

Proof. Suppose that $A \leq_p B$ and $B \in P$. Let $f(x)$ be a mapping reduction from A to B. The following program $a(x)$ tells whether $x \in A$ in polynomial time.

```
"{a(x):
   y = f(x)
   if y ∈ B
      return 1
   else
      return 0
}"
```

1. The program correctly tells whether $x \in A$ because $x \in A \leftrightarrow f(x) \in B$.

2. Because $f(x)$ is a polynomial-time mapping reduction, step

$$y = f(x)$$

can be done in polynomial time (say, n^k) in the length n of x. Since $B \in P$, test

$$y \in B$$

can be done in polynomial time (say, m^j) in the length m of y. But we need to know what that is in terms of n. How long can y be? Since $f(x)$ runs in time n^k, it only has time to write down n^k symbols. So $m \le n^k$ and testing whether $y \in B$ can be done in time $(n^k)^j = n^{kj}$. That is polynomial time in the length n of x.

Corollary 16.4. If $A \le_p B$ and $A \notin \mathrm{P}$ then $B \notin \mathrm{P}$.

Proof. Use Theorem 16.3 and tautology

$$(p \wedge q) \rightarrow r \;\leftrightarrow\; (p \wedge \neg r) \rightarrow \neg q.$$

◇

Theorem 16.5. If $A \le_p B$ and $B \le_p C$ then $A \le_p C$. That is, relation $\le_p$ is *transitive*.

Proof. Suppose that $f(x)$ is a polynomial-time mapping reduction from A to B and $g(y)$ is a polynomial-time mapping reduction from B to C. By the definition of a mapping reduction, for every x and y,

$$x \in A \quad \leftrightarrow \quad f(x) \in B \tag{1}$$

$$y \in B \quad \leftrightarrow \quad g(y) \in C \tag{2}$$

Define $h(x) = g(f(x))$. Notice that

$$\begin{aligned} x \in A \quad &\leftrightarrow \quad f(x) \in B && \text{by (1)} \\ &\leftrightarrow \quad g(f(x)) \in C && \text{by (2)} \\ &\leftrightarrow \quad h(x) \in C && \text{by the definition of } h(x) \end{aligned}$$

Also, $h(x)$ can be computed in polynomial time. If $f(x)$ is computable in time $O(n^k)$ and $g(y)$ is computable in time $O(n^j)$ then $h(x) = g(f(x))$ can be computed in time $O(n^{jk})$ by an argument similar to the one in the proof of Theorem 16.3.

16.3 Definition of an NP-Complete Problem

> **BIG IDEA:** We can identify problems that are among the most difficult problems that are in NP.

Suppose that you want to find the tallest person t in a room. The first requirement, of course, is that person t must be in the room. The second is that person t must be at least as tall as every other person in the room.

Similarly, a decision problem A is a hardest problem in NP if A is in NP and A is at least as hard as every other problem in NP. A hardest problem in NP is called an NP-complete problem.

Definition 16.6. Suppose that A is a language. Say that A is *NP-complete* if

(a) $A \in NP$

(b) For every language $X \in NP$, $X \leq_p A$.

Since A is in NP, the second condition says that A is as hard as *every* problem in NP, including A itself. That is okay: $A \leq_p A$ is clearly true. A is at least as hard as itself.

16.4 Consequences of NP-Completeness

It is not obvious that there exists an NP-complete problem. In Section 17, we will see some problems that are provably NP-complete. But right now, let's ask what NP-completeness tells us about a problem.

Our goal is to identify problems that are in NP but not in P. But nobody knows whether there exist *any* problems that are in NP that are not in P! Clearly, NP-completeness does not take us to our goal.

But suppose, for the sake of argument, that it turns out that P $\neq$ NP, and there is at least one language D in NP $-$ P. Also, suppose that problem E is NP-complete. Since $D \in$ NP, it must be the case that $D \leq_p E$. (*All* languages in NP polynomial-time reduce to an NP-complete problem.) Since $D \notin$ P, by Corollary 16.4, $E \notin P$. We have just proved the following.

Theorem 16.7. If P $\neq$ NP and E is NP-complete then $E \notin P$.

On the other hand, what if P = NP? By definition, an NP-complete problem is in NP, so if P = NP, then an NP-complete problem is also in P. That does not mean the problem is not NP-complete. It just means that NP-completeness is not interesting.

It is widely conjectured that P $\neq$ NP. But nobody knows if the conjecture is true.

Conjecture 16.8. P $\neq$ NP.

What would happen if someone finds a polynomial-time algorithm for an NP-complete problem? The following theorem tells you.

Theorem 16.8. If E is NP-complete and $E \in$ P then P = NP.

Proof. Suppose X is an arbitary problem in NP. Since E is NP-complete, $X \leq_p E$. By Theorem 16.3, since X polynomial-time reduces to a problem that is in P, X is also in P.

So, if the conditions stated in Theorem 16.8 are true, then every problem in NP is also in P. That is NP $\subseteq$ P. Since P $\subseteq$ NP (Theorem 15.13), P = NP.

$\diamondsuit$

So, if you are so inclined, you know what would be involved in proving that Conjecture 16.8 is wrong. Just find a polynomial-time algorithm for a problem that is known to be NP-complete. But be careful! Every year, a few people have tried to do exactly that. But their algorithms either do not run in polynomial time or do not work.

17 Examples of NP-Complete Problems

17.1 SAT

Section 15.4 defines the Satisfiability Problem for Propositional Logic (SATPL). We have seen that SATPL is in NP, and noticed in (Section 14) that SATPL appears to be difficult to solve.

Here, we look at a restriction of that problem to *clasual* propositional formulas. (The restriction can only make the problem easier, if anything.)

Definition 17.1. A *literal* is either a propositional variable or its negation. We will use lower-case letters such as x, y and z as propositional variables. Rather than writing $\neg y$ to indicate negated variable y, we write $\overline{y}$. Literal x is a *positive literal* and $\overline{y}$ is a *negative literal*.

Definition 17.2. A *clause* is a disjunction ($\vee$) of one or more literals. For example, $x \vee \overline{z} \vee y$ is a clause. A literal by itself is a clause with just one literal.

Definition 17.3. A *clausal formula* is a conjunction ($\wedge$) of one or more clauses. For example, $(x) \wedge (\overline{y} \vee z) \wedge (\overline{x} \vee y \vee \overline{z})$ is a clausal formula. There can be just one clause.

Definition 17.4. *SAT* is the following decision problem.

Input. A clausal propositional formula ϕ.
Question. Is ϕ satisfiable?

For example, is $(x) \wedge (\overline{y} \vee z) \wedge (\overline{x} \vee y \vee \overline{z})$ satisfiable? Of course: choose x, y and z all to be true. It is easy to check whether short propositional formulas are satisfiable. It is the long formulas that present difficulties!

Theorem 17.5. SAT $\in$ NP.

Proof. Theorem 15.5 shows that SATPL is in NP. But SAT is a restriction of SATPL. The evidence checker for SATPL also works for SAT.

$\diamondsuit$

Cook and Levin independently showed that SAT is NP-complete. The proof is too long for this course, so we will need to accept it as proved.

Theorem 17.6. (Cook/Levin Theorem) SAT is NP-complete.

That gives us evidence (but not a proof, since we don't know whether P $\neq$ NP) that there is no polynomial-time algorithm for SAT.

17.2 Proving NP-Completeness

SAT is proved NP-complete using a difficult kind of reduction called a *generic reduction*. If all you know about X is that $X \in$ NP, you can ask someone to give you an evidence checker for X. The generic reduction from X to SAT converts that evidence checker into a propositional formula. You can think of it as building a simple computer from logic gates that do nots, ands and ors.

But we don't need to do a generic reduction for every proof of NP-completeness.

Theorem 17.7. Suppose that language $B \in$ NP, A is NP-complete and $A \leq_p B$. Then B is NP-complete.

Proof. $B \in$ NP, so it suffices to show that $X \leq_p B$ for every language $X \in$ NP. Since A is NP-complete, we know that $X \leq_p A$ for every language $X \in$ NP. But $A \leq_p B$ and relation $\leq_p$ is transitive (Theorem 16.4), so $X \leq_p B$ for every language $X \in$ NP.

$\diamondsuit$

17.3 3-SAT

We can restrict the satisfiability problem further.

Definition 17.8. A propositional formula is in *3-clausal form* if it is in clausal form and has exactly 3 literals per clause. For example, $(x \vee y \vee z) \wedge (\overline{y} \vee z \vee \overline{w})$ is in 3-clausal form. A propositional formula in 3-clausal form is called a *3-clausal propositional formula*.

Definition 17.9. *3-SAT* is the following decision problem.

Input. A 3-clausal propositional formula ϕ.
Question. Is ϕ satisfiable?

Theorem 17.10. 3-SAT is NP-complete.

Proof. Clearly, 3-SAT is in NP. (Use the same evidence checker as for SATPL.) So, by Theorem 17.7, it suffices to reduce SAT to 3-SAT.

We need a polynomial-time algorithm that takes a clausal formula ϕ and builds a 3-clausal formula ϕ' such that ϕ is satisfiable if and only if ϕ' is satisfiable. Our algorithm will convert each clause of ϕ separately.

Clauses that already have 3 literals are left alone. Clauses with fewer than 3 literals are easy to deal with by duplicating one or more of the literals. For example, clause $(A \vee B)$ is equivalent to $(A \vee A \vee B)$.

That only leaves *long clauses*, which have more than 3 literals. As long as there is at least one long clause, we find one with $n > 3$ literals and replace it by a clause that has $n - 1$ literals, plus a clause with 3 literals. By repeating that, we can get rid of all of the long clauses. It is just a matter of ensuring that each step preserves satisfiability.

Suppose that ϕ contains clause

$$C = (\ell_1 \vee \ell_2 \vee \cdots \vee \ell_n)$$

where $n > 3$. Create a new variable u and replace C by pair of clauses

$$C' = (\ell_1 \vee \cdots \vee \ell_{n-2} \vee u) \wedge (\overline{u} \vee \ell_{n-1} \vee \ell_n)$$

yielding new formula ϕ_1. We need to show that ϕ_1 is satisfiable if and only if ϕ is satisfiable, showing that the modification of ϕ preserves satisfiability.

Claim 1. If ϕ_1 is satisfiable then ϕ is satisfiable. In fact, every truth-value assignment that satisfies ϕ_1 also satisfies ϕ.

Proof of Claim 1. Suppose ϕ_1 is satisfiable. Choose a truth-value assignment a that makes ϕ_1 true. That assignment must make all of the clauses other than C in ϕ true, since those clauses also occur in ϕ_1. We just need to argue that assignment a also makes clause C true. Be sure to notice that, because a makes all clauses in ϕ_1 true, it makes both clauses in C' true.

Suppose $a(u) = \mathrm{F}$. Then, because a makes clause $(\ell_1 \vee \cdots \vee \ell_{n-2} \vee u)$ true, a must make at least one of $\ell_1, \ldots, \ell_{n-2}$ true. But that means a makes clause C true.

Suppose that $a(u) = \mathrm{T}$. Then, because a makes clause $(\overline{u} \vee \ell_{n-1} \vee \ell_n)$ true, a makes at least one of ℓ_{n-1} and ℓ_n true. Again, a makes clause C true.

Claim 2. If ϕ is satisfiable then ϕ_1 is satisfiable.

Proof of Claim 2. Suppose that ϕ is satisfiable, and choose a truth-value assignment a that makes ϕ true. Clause C must have at least one true literal.

If a makes ℓ_i true where $i \le n-2$, then extend a by adding $u = \text{F}$. The new truth-value assignment makes clause $(\ell_1 \vee \cdots \vee \ell_{n-2} \vee u)$ true because ℓ_i is true, and it makes clause $(\overline{u} \vee \ell_{n-1} \vee \ell_n)$ true because $u = \text{F}$.

If a makes ℓ_i true where $i > n-2$, then extend a by adding $u = \text{T}$. You can check that both clauses of C' must be true.

♢

17.4 The Vertex Cover Problem

Recall from Section 15.4.2 that a vertex cover of a simple graph is a set C of vertices so every edge is incident on at least one vertex in C. The Vertex Cover Problem VCP is the following decision problem.

> **Input.** A simple graph G and a positive integer k.
> **Question.** Does there exist a vertex cover C of G where $|C| \le k$?

It is worth asking whether there is an obvious polynomial-time algorithm for VCP. One idea is to use a *greedy algorithm*, which tries to optimize globally by optimizing locally. Since we want to select as few vertices as possible to cover all of the edges, it makes sense to start by selecting a vertex with highest degree, since it covers as many edges as possible with the first pick. After that, remove the selected vertex and all of the edges that it covers, and repeat, again selecting a vertex with the highest degree.

That algorithm seems appealing, but does it work? Look at graph G_1 in Figure 17.1. The diagram shows a vertex cover of G_1 of size 5, but G_1 also has a vertex cover of size 4. (Can you find it?) G_1 has a vertex of degree 4, and all other vertices have degree 2 or 3. But the degree 4 is not part of any smallest vertex cover of G_1! If you are trying to determine whether G_1 has a vertex cover of size at most 4, you will be led astray by selecting the degree 4 vertex. Something is wrong with the greedy Vertex Cover algorithm.

It is tempting to try to patch the greedy algorithm. But is that worthwhile? The following theorem shows that it is a waste of time.

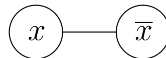
Theorem 17.11. VCP is NP-complete.

If the conjecture P $\neq$ NP is true then there does not exist a polynomial-time algorithm for VCP. Even if the conjecture is wrong, finding a polynomial-time algorithm for VCP is as difficult as proving that P = NP, since the existence of such an algorithm implies P = NP.

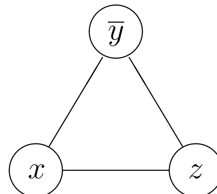
Proof of Theorem 17.11 Section 15.4.2 shows that VCP is in NP. We only need to reduce a known NP-complete problem to VCP. Let's show that 3-SAT $\leq_p$ VCP.

We need a polynomial-time algorithm that takes a propositional formula ϕ in 3-clausal form and builds a pair consisting of a simple graph G and a positive integer k, where ϕ is satisfiable if and only if G has a vertex cover of size at most k.

The first step is construction of G. There are three parts. Part 1 consists of a pair of vertices for each variable that occurs in ϕ, which we call a *vertex gadget*. If x is a variable, add the following, where one vertex is labeled x and the other is labeled $\overline{x}$.



Part 2 consists of three vertices for each clause of ϕ, all connected to one another and labeled by the three literals in the clause, which we call a *clause gadget*. For clause $(x \vee \overline{y} \vee z)$, we add

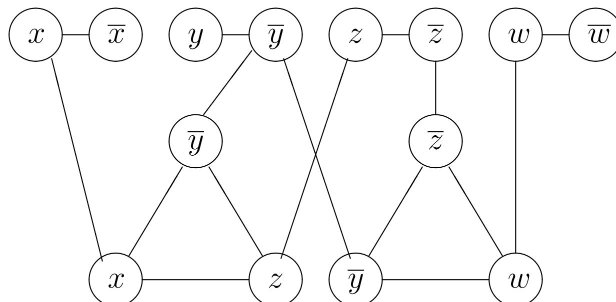


Part 3 does not add any vertices, but adds edges between part 1 vertices and part 2 vertices. Specifically, each part 2 vertex is connected to the part 1 vertex that has the same label.

Here is an example. Suppose

$$\phi = (x \vee \overline{y} \vee z) \wedge (\overline{y} \vee \overline{z} \vee w).$$

Then graph G looks like this:



That finishes the description of G. Suppose that ϕ has v variables and c clauses. Any vertex cover of G will need to have size at least $v + 2c$, one to cover each vertex gadget and two to cover each clause gadget. Let's choose $k = v + 2c$, not leaving any room for extra vertices in the vertex cover.

We need to prove that G has a vertex cover of size $v + 2c$ if and only if ϕ is satisfiable. Let's do that in two parts, proving the 'if' and the 'only if' parts separately.

Claim 1. If ϕ is satisfiable then G has a vertex cover of size $v + 2c$.

Proof of Claim 1. Suppose ϕ is satisfiable, and let a be a truth-value assignment that makes ϕ true. Here is how to select a vertex cover of G of size $v + 2c$.

(a) For each variable x, if $a(x) = \text{T}$ then select the vertex gadget-vertex labeled x. Otherwise, select the vertex-gadget vertex labeled $\overline{x}$. That puts one vertex for each vertex gadget in the vertex cover, which covers the edges within vertex gadgets.

(b) For each clause $C = (\ell_1 \vee \ell_2 \vee \ell_3)$, find a literal ℓ_i that truth-value assignment a makes true. Select the clause gadget vertices that correspond to the other two literals, leaving the vertex labeled ℓ_i unselected. That covers all edges within clause gadgets.

There is no room to select any more vertices, so the part 3 edges between clause gadgets and vertex gadgets need to be covered by the vertices that have already been selected. The unselected vertex u in a clause gadget corresponds to a true literal ℓ_i (under truth-value assignment a). A part 3 edge connects u to a vertex-gadget vertex v labeled ℓ_i, and, since ℓ_i is true, vertex v was

selected, and edge $\{u, v\}$ is covered by v. No more vertices need to be added.

Claim 2. If G has a vertex cover of size $v + 2c$ then ϕ is satisfiable.

Proof of Claim 2. Suppose G has a vertex cover S of size $v+2c$. We know that S must select exactly one vertex from each vertex gadget and exactly two vertices from each clause gadget. Define truth-value assignment a so that $a(x) = \text{T}$ if the vertex-gadget vertex labeled x is in S, and choose $a(x) = \text{F}$ if the vertex-gadget vertex labeled $\overline{x}$ is in S.

Consider a clause gadget C. It must have one vertex u that is not in vertex cover S. Suppose u is labeled by literal ℓ. There is an edge in G between u and a vertex-gadget vertex v that is also labeled ℓ. Since S is required to cover all edges, and S does not contain u, S must contain v.

But truth-value assignment a has been defined so that literal ℓ is true; that is, if v is labeled x then $a(x) = \text{T}$, and if v is labeled $\overline{x}$ then $a(x) = \text{F}$, making $\overline{x}$ true. Therefore, the clause of ϕ that corresponds to clause gadget C has a true literal, namely ℓ.

The two claims show that the algorithm described above is a mapping reduction from 3-SAT to VCP. (It should be obvious that the algorithm runs in polynomial time.)

$\diamondsuit$

17.5 The Independent Set Problem

Definition 17.12. Suppose $G = (V, E)$ is a simple graph. An *independent set* of G is a set $S \subseteq V$ such that no two members of S are connected by an edge. That is, if u and v are different members of S, then$\{u, v\} \notin E$.

Definition 17.13. The *Independent Set Problem* (ISP) is the following decision problem.

Input. A simple graph $G = (V, E)$ and a positive integer k.
Question. Does G have an independent set of size at least k?

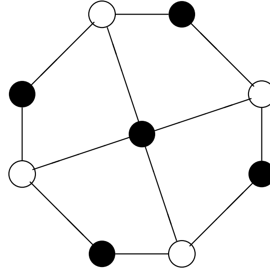


Figure 17.1 Graph G_1. The solid vertices are an independent set of G_1 and the empty circles are a vertex cover of G_1.

Look at graph G_1 in Figure 17.1. Some vertices are solid black and some are circles. Notice that the solid vertices are a vertex cover of G and the empty circles are an indenpendent set of G. Is that a coincidence? Think about it.

Theorem 17.14. Suppose $G = (V, E)$ is a simple graph and $S \subseteq V$. S is a vertex cover of G if and only if $\overline{S}$ is an independent set of G.

Proof. Suppose that $G = (V, E)$. Saying that S is a vertex cover of G is equivalent to the following logical statement.

$$\forall u \forall v(\{u, v\} \in E \to (u \in S \vee v \in S)).$$

Using the law of the contrapositive, that is equivalent to

$$\forall u \forall v(\neg(u \in S \vee v \in S) \to \{u, v\} \notin E).$$

Using DeMorgan's law and the definition of $\overline{S}$, that is equivalent to

$$\forall u \forall v((u \in \overline{S} \wedge v \in \overline{S}) \to \{u, v\} \notin E).$$

That is exactly what it means for $\overline{S}$ to be an independent set of G.

♢

Theorem 17.15. VCP $\le_p$ ISP.

Proof. Suppose that G has n vertices. For any set of vertices S of G, $|\overline{S}| = n - |S|$. That means $f(G, k) = (G, n-k)$ is a polynomial-time reduction

from VCP to ISP, since

$$
\begin{aligned}
(G,k) \in \text{VCP} &\leftrightarrow G \text{ has a vertex cover of size at most } k\\
&\leftrightarrow G \text{ has an independent set of size at least } n-k\\
&\leftrightarrow (G, n-k) \in \text{ISP}
\end{aligned}
$$

$\diamondsuit$

Corollary 17.16. ISP is NP-complete.

Proof. It is clear that ISP $\in$ NP. Theorem 17.15 shows that known NP-complete problem VCP polynomial-time reduces to ISP.

$\diamondsuit$

17.6 The Clique Problem

Another NP-complete problem about graphs is the Clique Problem.

Definition 17.17. Suppose that $G = (V, E)$ is a simple graph. A set $S \subseteq V$ is a *clique* if every pair of vertices in S are adjacent. That is, S is a clique if for all pairs of different vertices u and v in S, $\{u, v\} \in E$.

Definition 17.18. The *Clique Problem* (CP) is the following decision problem.

Input. A simple graph G and a positive integer k.
Question. Does G have a clique of size at least k?

Definition 17.19. Suppose $G = (V, E)$ is a simple graph. Then $\overline{G} = (V, \overline{E})$ is the complement of G, formed by complementing the set of edges. That is, $\overline{G}$ has an edge between different vertices u and v if and only if G does not have an edge between u and v.

The following Theorem 17.20 is immediate from the definitions of independent sets and cliques.

Theorem 17.20. Suppose $G = (V, E)$ is a simple graph. S is an independent set of G if and only if S is a clique of $\overline{G}$.

Theorem 17.21. ISP $\le_p$ CP

Proof. By Theorem 17.20, function $f(G, k) = (\overline{G}, k)$ is a polynomial-time reduction from ISP to CP.

$\diamondsuit$

17.7 The Subset Sum Problem

The Subset Sum Problem is a generalization of the Partition Problem that we looked in Section 15.

Definition 17.22. The *Subset Sum Problem* (SSP) is the following decision problem.

Input. A list $x_1, \ldots, x_n$ of positive integers and a positive integer K.
Question. Does there exist an index set $I \subseteq \{1, \ldots, n\}$ so that

$$\sum_{i \in I} x_i = K?$$

We will show that SSP is NP-complete by showing that SSP is in NP and that 3-SAT $\le_p$ SSP.

Theorem 17.23. SSP $\in$ NP.

Proof. The question in the definition of SSP is a question of existence. That suggests using I, the thing whose existence is questioned, as the evidence. Here is a polynomial-time evidence checker for SSP.

Evidence checker for SSP	
Input.	List $x_1, \ldots, x_n$ of positive integers and positive integer K
Evidence.	Index set $I \subseteq \{1, \ldots, n\}$
Requirement.	$\sum_{i \in I} x_i = K$?

$\diamondsuit$

Theorem 17.24. 3-SAT $\le_p$ SSP.

	w	z	y	x	c_1	c_2	c_3
N_x:				1	1	0	0
$N_{\overline{x}}$:				1	0	0	1
N_y:			1	0	1	0	0
$N_{\overline{y}}$:			1	0	0	1	0
N_z:		1	0	0	0	1	1
$N_{\overline{z}}$:		1	0	0	1	0	0
N_w:	1	0	0	0	0	0	1
$N_{\overline{w}}$:	1	0	0	0	0	1	0
$P_{1,1}$:					1	0	0
$P_{1,2}$:					1	0	0
$P_{2,1}$:					0	1	0
$P_{2,2}$:					0	1	0
$P_{3,1}$:					0	0	1
$P_{3,2}$:					0	0	1
K:	1	1	1	1	3	3	3

Figure 17.2. List L consists of N_x and $N_{\overline{x}}$ for each variable x plus $P_{i,1}$ and $P_{i,2}$ for each clause c_i. Numbers are written in base 10. Notice that the sum can never involve a carry since there are no more than five 1s in any column.

Proof. Like the proof of Theorem 17.11, showing that 3-SAT $\leq_p$ VCP, this proof requires some thought and some gadgetry. A polynomial-time reduction from 3-SAT to SSP is a polynomial-time computable function $f(\phi) = (L, K)$ where ϕ is a propositional formula in 3-clausal form, $L = x_1$, $\ldots$, x_n is a list of positive integers and K is a positive integer so that ϕ is satisifiable if and only if $(L, K) \in$ SSP.

Writing a program for the reduction is not very informative. It is much easier to understand the reduction from an example. Suppose that ϕ is

$$(x \vee y \vee \overline{z}) \wedge (\overline{y} \vee z \vee \overline{w}) \wedge (w \vee \overline{x} \vee z)$$

with clauses c_1, c_2 and c_3. The result (L,K) of $f(\phi)$ is shown in Figure 17.2. List L is broken into two parts.

Part 1 of list L has two numbers N_x and $N_{\overline{x}}$ for each variable x. Think of those numbers written in base 10, with each number having two sections, the

variable section and the *clause section*. The variable section has a column for each variable and the clause section has a column for each clause.

1. In the variable section, N_x and $N_{\overline{x}}$ each have a 1 in the column for x, with all other digits in the variable section being 0.

2. In the clause section, N_x has a 1 in column c_i if x occurs in clause c_i, and it has a 0 in column c_i otherwise. Similarly, $N_{\overline{x}}$ has a 1 in column c_i if $\overline{x}$ occurs in clause c_i, and a 0 in column c_i otherwise.

Part 2 of list L has two numbers $P_{i,1}$ and $P_{i,2}$ for each clause c_i, which both contain only a 1 in the column that corresponds to c_i, as shown in Figure 17.2. They are used as *padding*.

We need to show that ϕ is satisfiable if and only if there is a way to select numbers from list L whose sum is exactly K. As before, we prove the 'if' part and the 'only if' part separately.

Claim 1. If ϕ is satisfiable then there is a way to select numbers from list L whose sum is K.

Proof of Claim 1. Suppose that a is a truth-value assignment that makes ϕ true. It tells which numbers to select to make a sum of K. First, select a true literal from each clause. If literal x is selected, put N_x into the list of selected numbers. If literal $\overline{x}$ is selected, put $N_{\overline{x}}$ into the list of selected numbers. If neither x nor $\overline{x}$ is selected, it does not matter; put N_x into the list of selected numbers.

Notice that the sum of the selected numbers has exactly one 1 in each column of the variable section, so the variable section of the sum K is correct.

Now we need to make sure the section of K consisting of 3's is correct. Because each clause contains a true literal, there must be at least one 1 in each clause column. But the total number of 1s in a single clause column in part 1 can be at most 3 since each clause contains 3 literals. If a clause column has one 1, then select both of the padding (part 2) numbers for that column to make a total of exactly 3. If there are two 1's, select one of the padding numbers. If there are three 1's, do not select any of the padding numbers for that clause.

The sum of the selected numbers is exactly K.

Claim 2. If it is possible to select numbers from list L whose sum is K then ϕ is satisfiable.

Proof of Claim 2. Because each variable column must sum to 1, exactly one of N_x and $N_{\overline{x}}$ must have been chosen for each variable x. Define truth-value assignment a so that $a(x) = \text{T}$ if N_x is selected and $a(x) = \text{F}$ if $N_{\overline{x}}$ is selected.

The selected numbers must sum to 3 in each clause column. At most two 1s in column i can come from padding numbers. The third must come from N_x, where x occurs in clause c_i, or from $N_{\overline{x}}$ where $\overline{x}$ occurs in clause c_i. That means c_i contains a true literal under truth-value assignment a.

The two claims show that the algorithm described above is a mapping reduction from 3-SAT to SSP. It should be obvious that the algorithm runs in polynomial time.

$\diamondsuit$

17.8 Graph Coloring Problems

Let's look at some known NP-complete problems without proving them NP-complete.

Definition 17.25. Suppose that G is a simple graph and k is a positive integer. Say that G is *k-colorable* if it is possible to color each vertex of G with one of k colors so that no two adjacent vertices have the same color.

Definition 17.26. The *Graph Coloring Problem* is the following decision problem.

> **Input.** A simple graph G and a positive integer k.
> **Question.** Is G k-colorable?

The Graph Coloring Problem is clearly in NP. The question asks whether *there exists* a way to color the vertices of G so that no two adjacent vertices have the same color. The obvious evidence to request is the coloring.

Evidence checker for Graph Coloring	
Input	Simple graph G and positive integer k.
Evidence	Assignment A of one of k colors to each vertex of G.
Requirement	Every edge of G connects two vertices that are assigned different colors in color assignment A.

If you try to color some graphs by hand, you can get an idea of how difficult Graph Coloring can be. The Graph Coloring Problem is known to be NP-complete. In fact, it is NP-complete even if k is fixed at 3.

Definition 17.27. The *3-Coloring Problem* is the following decision problem.

Input. A simple graph G.
Question. Is G 3-colorable?

Graph Coloring is so difficult, it can even be restricted further and remain NP-complete. A graph is *planar* if it can be drawn in the plane (on a piece of paper, if you like) so that no two edges cross one another.

Definition 17.28. The *Planar 3-Coloring Problem* is the following decision problem.

Input. A planar simple graph G.
Question. Is G 3-colorable?

The Planar 3-Coloring Problem is NP-complete. But that does not mean that all graph coloring problems are NP-complete. For example, 2-coloring is easy. (Try an example.) Also, if G is known to be a planar graph, then 4-coloring is trivial: the answer is always yes, by the following.

Theorem 17.29. (The 4-Color Theorem) Every planar graph is 4-colorable.

17.9 Hamilton Cycles and Hamilton Paths

Definition 17.30. Suppose that G is a simple graph. A *simple cycle* in G is a cycle that does not contain any vertex more than once. A *Hamilton Cycle*

is a simple cycle that contains every vertex.

Not every graph has a Hamilton cycle. You should be able to find a graph that has a Hamilton cycle and another that does not.

Definition 17.31. The *Hamilton Cycle Problem* is the following decision problem.

Input. A simple graph G.
Question. Does G have a Hamilton cycle?

Imagine that G has been drawn on paper (possibly with edges crossing). The Hamilton Cycle Problem asks whether it is possible to draw a cycle, following the edges, that hits every vertex exactly once, without lifting your pencil off the paper.

It is easy to show that the Hamilton Cycle Problem is in NP. The obvious evidence is a Hamilton cycle.

Evidence checker for Hamilton Cycle	
Input	Simple graph G with n vertices.
Evidence	Sequence $v_1, \ldots, v_n$ of vertices of G.
Requirement	$v_1, \ldots, v_{n-1}$ contains every vertex exactly once, $v_1 = v_n$ and for $i = 1, \ldots, n-1$, $\{v_i, v_{i+1}\}$ is an edge of G.

The Hamilton Cycle Problem is known to be NP-complete. A related problem, also NP-complete, is the Hamilton Path Problem.

Definition 17.32. Suppose that G is a simple graph. A *simple path* in G is a path that does not contain any vertex more than once. A *Hamilton Path* is a simple path that contains every vertex (exactly once).

Definition 17.33. The *Hamilton Path Problem* is the following decision problem.

Input. A simple graph G.
Question. Does G have a Hamilton path?

17.9.1 Euler Cycles

A problem that is at least superficially related to the Hamilton Cycle Problem is the Euler Cycle Problem. (Leonard Euler's last name is pronounced "Oiler".)

Definition 17.34. Suppose that G is a simple graph. An *Euler Cycle* in G is a cycle that uses each *edge* exactly once. (The cycle can contain a particular *vertex* several times.)

Not every graph has an Euler cycle. You should be able to find a graph that has an Euler cycle and another that does not.

Definition 17.35. The *Euler Cycle Problem* is the following decision problem.

Input. A simple graph G.
Question. Does G have an Euler cycle?

How difficult is it to determine whether a graph contains an Euler cycle? It is easy to see that the Euler Cycle Problem is in NP.

Evidence checker for Euler Cycle	
Input	Simple graph G with n vertices.
Evidence	Sequence $v_1, \ldots, v_n$ of vertices of G.
Requirement	$v_1 = v_n$, for $i = 1, \ldots, n-1$, $\{v_i, v_{i+1}\}$ is an edge of G (so $v_1, \ldots, v_n$ is a cycle) and cycle $v_1, \ldots, v_n$ uses each edge exactly once.

So we have an *upper bound* on the difficulty of solving the Euler Cycle Problem: it is in NP, so it is, to within a polynomial, no worse than SAT. But that is not a *lower bound*. It might be that the Euler Cycle Problem is easy to solve.

And in fact, it is! Graph G has an Euler cycle if and only if every vertex has even degree. That is easy to check. Not only is the Euler Cycle Problem in P, but it is solvable in time $O(n)$.

There is a lesson in that. You cannot inspect a problem and conclude, based on its similarity to another problem, that it is an easy or a difficult problem.

To show that a problem is in P, find a polynomial-time algorithm for it, and *make sure that the algorithm works*. To show that a problem is NP-complete, show that it is in NP and that a known NP-complete problem reduces to it in polynomial time. There are no shortcuts.

18 Beyond NP

The theory of NP-completess is a bedrock of computer science because there are so many NP-complete problems, and they crop of everywhere. There are NP-complete problems from mathematics, the theory of databases, the theory of compilers and even from politics.

But even though NP-completeness is central, there is more to the world than that. This section looks at some other classes of problems.

18.1 Co-NP and the Validity Problem

We started looking at difficult problems in Section 14.3 with the Validity Problem for Propositional Logic (VALIDPL). But we have not said anything more about it yet. We have not shown that it is NP-complete, nor have we shown that it is in P.

There is a good reason for that. The validity problem is conjectured to be neither in P nor NP-complete. That is a consequence of the asymmetry of NP: if $A \in \text{NP}$, then there are short, easily checkable proofs that things are in A, but there is no requirement that there are short, easily checkable proofs that things are *not* in A.

But VALIDPL has short, easily checkable proofs of *nonmembership.* To show that ϕ is not valid, show that $\neg\phi$ is satisfiable by finding a truth-value assignment that makes $\neg\phi$ true.

Let's define $\overline{\text{SATPL}}$ to be the set of propositional formulas that are not satisfiable. Then $f(\phi) = \neg\phi$ is a polynomial-time reduction from VALIDPL to $\overline{\text{SATPL}}$. The same function is a polynomial-time reduction from $\overline{SATPL}$ to VALIDPL. So VALIDPL is equivalent in difficulty to $\overline{\text{SATPL}}$.

To deal with VALIDPL, we need a class of languages that are complements of languages that are in NP:

$$\text{Co-NP} = \{X \mid \overline{X} \in \text{NP}\}.$$

Pay close attention to the definition of Co-NP. Co-NP is not the complement of NP.

Now we can define the class of hardest problems in Co-NP in a way that is analogous to the way NP-complete problems are defined. Say that a language L is *Co-NP-complete* provided $L \in$ Co-NP and $X \leq_p L$ for every language $X \in$ Co-NP.

It is easy to show that A is NP-complete if and only if $\overline{A}$ is Co-NP-complete, for every language A. For example, since VALIDPL is equivalent to $\overline{SATPL}$, VALIDPL is Co-NP-complete.

It is also easy to show that, if P $\neq$ NP, then a Co-NP-complete problem has no polynomial-time algorithm. (Imagine, for example, a polynomial-time Turing-reduction from SATPL to VALIDPL. That shows that VALIDPL $\in$ P $\rightarrow$ SATPL $\in P$, or, by taking the contrapositive, SATPL $\notin P \rightarrow$ VALIDPL $\notin P$.)

18.2 NP Intersect Co-NP and Factoring

We know that P $\subseteq$ NP. By symmetry, P $\subset$ Co-NP. So P $\subset$ NP $\cap$ Co-NP.

In Section 13.3 we saw co-partially computable languages and found that the intersection of the class of partially computable sets with the class of co-partially computable sets is exactly the class of computable sets. An obvious question is whether an analogous thing happens here: Is P = NP $\cap$ Co-NP?

Surprisingly, it is conjectured that P $\neq$ NP $\cap$ Co-NP.

Conjecture 18.1 P $\neq$ (NP $\cap$ Co-NP).

What would lead people to make Conjecture 18.1? There must be some decision problem that is conjectured to be in NP$\cap$Co-NP but not in P. And there is: factoring integers. Quick, what are the prime factors of 109,938,432,277?

Actually, the problem of factoring a given integer cannot be in NP $\cap$ Co-NP because it is not a decision problem; the result is a list of factors. But there is a decision problem that is has the same level of difficulty.

Definition 18.2. FACTOR is the following decision problem.

Input. Two positive integers x and k.
Question. Does there exist a factor y of x where $1 < y < k$?

If you have a polynomial-time algorithm that finds the factors of an integer then it is easy to decide FACTOR. And if you have a polynomial-time

algorithm for FACTOR then you can find the smallest factor of an integer using binary search. Having found the smallest factor, you divide x by that factor and continue finding factors, stopping when the number that you have is prime. (As mentioned in Section 14, there is a known algorithm that determines whether a given integer is prime in polynomial time.)

There is no known polynomial-time algorithm for FACTOR and FACTOR is conjectured to be in $NP \cap$ Co-NP but not in P.

18.3 Public Key Cryptograpy

Cryptographic systems are based on keys. To encipher a message you use the encipher key, and to decipher a message you use the associated decipher key. A person who has the decipher key is said to *decipher* a message. A person who attempts to do the same job without the benefit of the decipher key is said to *decrypt* the message.

Traditional cryptography is based on the idea that someone attempting to decrypt an enciphered message does not have enough *information* to do so. The standard traditional cryptosystem is a one-time pad, where a randomly chosen sequence of bits called the *pad* is used as a key. To encipher a message (a sequence of bits), you do a bitwise exclusive-or of the message with the pad. To decipher a message, you also do a bitwise exclusive-or of the enciphered message with the pad, which gives the original message back. The pad must only be used once. Using it more than once risks giving away information to an adversary.

Public key cryptography takes a different viewpoint. The encipher and decipher keys are different, and the strength of the system is based on the idea that someone trying to decrypt a message does not have enough *time* to do that. And that depends on the problem of decrypting a message being computationally difficult.

Definition 18.3. A public key cryptosystem is described by two functions $E(k, x)$ and $D(j, y)$ where k is a public encipher key and j is a private decipher key. Functions $E(k, x)$ and $D(j, y)$ must have the following properties.

1. For every x in a limited range, $D(j, E(k, x)) = x$ and $E(k, D(j, x)) = x$. That is, deciphering an enciphered message gives the original message, and enciphering a deciphered message also gives the original message.

2. There is a polynomial-time algorithm to compute $E(k, x)$ and another to compute $D(j, y)$.

3. The *decryption function* $C(k, y)$ is defined to take the encipher key k and an enciphered message y and yield a value x such that $E(k, x) = y$; $C(k, y)$ decrypts without the benefit of j; it is only told k. There should be no polynomial-time algorithm that computes $C(k, y)$.

   Because the encipher key k is public, we write $C(y)$ rather than $C(k, y)$. That slightly simplifies what follows.

The strength of a public key cryptosystem is tied to the (at least apparent) computational difficult of computing $C(y)$.

Several public key cryptosystems are known, and it is not our concern here to describe one. Rather, let's ask whether a public-key cryptosystem exists. Assume that function pair ($E(k, x)$, $D(j, y)$) is such a cryptosystem. For simplicity, assume that messages (x and y) are integers. Text can always be encoded using integers.

Consider the following decision problem DECRYPT determined by the decryption function $C(y)$.

$$\text{DECRYPT} = \{(y, i) \mid C(y) < i\}.$$

DECRYPT must be in NP. As evidence, use the decipher key j. First compute $x = D(j, y)$, then compute $z = E(k, x)$. Accept j as evidence that $(y, i) \in$ DECRYPT provided $z = y$ and $x < i$. Requirement $z = y$ tells you that j is the correct decipher key and requirement $x < i$ tells you that $C(y) < i$.

DECRYPT must also be in Co-NP. The complement of DECRYPT is equivalent to language $\{(y, i) \mid D(y) \geq i\}$, and a similar evidence checker works for that.

So DECRYPT is in NP $\cap$ Co-NP. But if DECRYPT is in P then there is a polynomial-time algorithm to compute $C(y)$. Simply use binary search to search for the smallest i such that $D(y) < i$. Then $C(y) = i - 1$. That leads to the following conclusion.

Theorem 18.4. A public key cryptosystem can only exist if P $\neq$ NP $\cap$ Co-NP.

It is no accident that public key cryptosystems are based on factoring or on other problems that are in NP $\cap$ Co-NP.

18.4 Polynomial Space

PSPACE is the class of all decision problems that can be solved using $O(n^k)$ bits of memory for some fixed k, where n is the length of the input.

It is known that NP $\subseteq$ PSPACE and Co-NP $\subseteq$ PSPACE, and it is conjectured that NP $\neq$ PSPACE (and Co-NP $\neq$ PSPACE). Polynomial space allows a lot of room for computations. Typical exponential-time algorithms only use a polynomial amount of memory.

A PSPACE-complete problem is one of the hardest problems in PSPACE.

Definition 18.5. A decision problem A if *PSPACE-complete* if

(a) A is in PSPACE and

(b) $X \leq_p A$ for every problem $X \in$ PSPACE.

Several PSPACE-complete problems are related to two-person games. An example is *Generalized Checkers*, defined as follows.

> **Input.** A placement of red and black kings on an $n \times n$ checkerboard.
> **Question.** Assuming that it is red's move, does red have a winning strategy from the given configuration?

Geography is a game that children can play without any props. A child thinks of the name of a country (or other chosen category). If the first child selects Sweden, then the next child must select a country name that begins with N, the last letter of Sweden. Suppose that child chooses Nepal. Now the next player must choose a country name that starts with L. Countries cannot be reused, and the first child who cannot think of a country name loses.

There is a version of Geography that is played on a directed graph. A vertex is selected as the start vertex s. The first player selects a vertex u where there is a directed edge from s to u. The second player selects a vertex v where there is a directed edge from u to v. Play alternates. No vertex that was previously selected can be selected again.

The *Generalize Geography* decision problem is as follows.

> **Input.** A directed graph G with a selected start vertex.
> **Question.** Does the first player have a winning strategy on the game of Geography played on G?

Generalize Geography is known to be PSPACE-complete.

In practice, PSPACE-complete problems appear very difficult to solve. Not only do they appear not to have polynomial-time algorithms, but they do not appear to have polynomial-time evidence checkers. Even if someone knows the answer to a particular input of PSPACE-complete problem, he or she cannot convince you that the answer is correct using a short, easy to check proof!

It is worth thinking about how you would write an evidence checker for Generalized Geography. What would the evidence be? The most obvious evidence that the first player has a winning strategy is the strategy. But that is really huge! It must not only say what the first player's first move is, it must say how the first player responds to each move of the second player. As the number of moves grows, the strategy grows exponentially in size. An exponential-size piece of evidence clearly cannot be checked in polynomial time.

But, even though PSPACE appears to be larger than NP, surprisingly, nobody knows whether PSPACE = P. Even the huge jump from polynomial time to polynomial space is not enough for us to demonstrate a separation. If you want to prove that P $\neq$ NP, you might warm up by proving that P $\neq$ PSPACE; that ought to be easier.

18.5 Exponential Time

Definition 18.6. *EXPTIME* is the class of decision problems that are solvable in time $O(2^{n^k})$ for some fixed k, where n is the length of the input.

It is known that PSPACE $\subseteq$ EXPTIME, and PSPACE is conjectured to be a proper subset of EXPTIME.

With EXPTIME, we finally have a provable separation! It is known that P $\neq EXPTIME$. So at least one of the subset relations P $\subseteq$ NP $\subseteq$ PSPACE

$\subseteq$ EXPTIME is surely a proper subset relation. Of course, they are all conjectured to be proper.

There is a notion of an EXPTIME-complete problem, defined in the usual way as a hardest problem in EXPTIME.

Definition 18.7. A decision problem A is *EXPTIME-complete* if

(a) A is in EXPTIME and

(b) $X \leq_p A$ for every problem $X \in$ EXPTIME.

There are EXPTIME-complete problems. There is a typed programming language called ML. The ML Type Checking problem is as follows.

Definition 18.9. The *ML Type Checking Problem* is the following decision problem.

Input. An ML program p.
Question. Is p well-typed (free of type errors)?

The ML Type Checking Problem is known to be EXPTIME-complete. Fortunately, ML programmers tend not to write programs that are difficult to type check. But if you want to, you can write a short ML program that will bring an ML compiler to its knees; in practice, the memory requirements overwhelm the compiler, and it gives up.