3 Theorems and Proofs

A *theorem* is any mathematical statement, such as a formula of first-order logic, that has been proved true or is about to be proved true. (When you are about to prove a theorem, you call it theorem as a way of promising that a proof is about to be produced.)

BIG IDEA: Doing proofs teaches you to reason carefully and to present your ideas precisely. Writing computer software requires you to reason carefully and to present your ideas (in the form of a computer program) precisely. People who can do proofs are prepared for the rigors of software development.

3.1 What Is a Proof?

There are many different precise definitions of a proof. But most mathematicians accept an informal definition: a proof is a clear and unambiguous argument that a mathematical statement is true and that any sufficiently knowledgeable person can check. The key is that a reader must be able to check that each step in the proof is correct.

Students who are just learning to do proofs make many different kinds of mistakes, but most fall into one of the following two categories.

1. **The student does not check his or her own work.** The reason for this can vary from lack of time to lack of understanding to fear of failure.

   Errors in elementary algebraic manipulations are surprisingly common. Simply looking at your work with a critical eye will usually suffice to find those errors.

   When you do a proof, it is a good idea to look at proofs of similar statements that you have already seen, and to modify those proofs to work for your current goal. But be cautious! It is easy to end up with something that makes no sense. Check the modified proof carefully with a critical eye.

The student who has a lack of understanding cannot check the proof. The only remedy is to gain the necessary understanding.

The student who is afraid of failure will not check the proof out of fear that it might turn out to be incorrect. That can be cured by adopting a policy of checking everything that you write and fixing it when you encounter an error. Take pride in your work and care whether it is right.

Regardless of your reason for not checking your proof, you can be sure that an unchecked proof is incorrect, for the same reasons that an untested computer program does not work. You will need to find a way to motivate yourself to check your proofs carefully.

2. Mathematics relies on precise definitions. When you do a proof, it is essential for you to use definitions wherever appropriate. **Students often get stuck in a proof because they have forgotten to use definitions**. Any time you cannot see how to proceed, ask yourself if using a definition will help. We will see examples of that.

3.2 Deus ex Machina

Deus ex machina is latin for "god from the machine." In ancient plays, there were certain rules that the playwright was required to follow. (Modern movies have similar constraints. You are not allowed to kill off the hero, for example, especially when the hero is played by a popular actor.) Sometimes the playwright got his characters in a real bind, and he could not see how to get them out of it. The solution was to have a platform rise up with an actor on it dressed as a god. The actor would wave his arms and fix everything.

I have frequently found that students resort to deus ex machina in their proofs. Having written part of a proof, the student finds that he or she cannot make further progress. So he or she simply writes the goal and claims that it has been proved. Voila!

I can only guess that a student does that in the hope of getting partial credit for the part of the "proof" before pulling the goal out of the air. But pulling the goal out of the air will *decrease* your score. It is better to admit that you are stuck. Even better, back up and try a different approach, or check your algebra; with correct algebra, you might not actually be in a bind at all.

3.3 Forward Proofs

A *forward proof* reasons from what you know to what you can conclude. The proof accumulates knowledge and (named) values until it reaches a point where the goal is known. Each new conclusion can rely on prior knowledge or conclusions.

You have probably been taught a different approach in an algebra class. In a *backwards proof*, you write down what you want to show and then perform some manipulations on it, working backwards to a statement that you already know is true, such as $x = x$.

In this class, we will do forward proofs, with small excursions that typically convert a goal into an equivalent goal, followed by a proof of the equivalent goal. **I expect you to use forward proofs as well.** At least for this class, put aside the backwards proofs that you have learned in algebra.

In this section, I do proofs at two different levels of detail. The first proof of a theorem works in small steps and shows everything that you know after each step. The second proof of the same theorem is more typical of what you would write, and what I want to see from you.

3.4 Some Definitions

Definition 3.1. Integer n is *even* if there exists an integer m such that $n = 2m$. For example, 6 is even because $6 = (2)(3)$.

Definition 3.2. Integer n is *odd* if there exists an integer m such that $n = 2m + 1$. We will also make use of the fact that, for every n, n is odd if and only if n is not even.

Definition 3.3. Integer n is a *perfect square* if there exists an integer m such that $n = m^2$.

Definition 3.4. Real number x is *rational* if there exist integers n and m where $m \neq 0$ such that $x = n/m$.

3.5 Proving A Implies B

You typically prove an implication by *direct proof*: **To prove $A \to B$, assume that A is true and show that B is true.** That is, add A to your knowledge. Then prove B.

Example Theorem 3.5. If n is even then n^2 is even.

Detailed Proof.

1. Suppose that n is even.

Known variables:	n
Know:	n is even.
Goal:	n^2 is even.

2. By the definition of an even integer, there exists an integer m such that $n = 2m$.

Known variables:	n, m
Know:	n is even.
Know:	$n = 2m$.
Goal:	n^2 is even.

3. Since $n = 2m$, $n^2 = (2m)^2 = 4m^2 = 2(2m^2)$.

Known variables:	n, m
Know:	n is even.
Know:	$n = 2m$.
Know:	$n^2 = 2(2m^2)$.
Goal:	n^2 is even.

4. So $n^2 = 2(x)$ where $x = 2m^2$. Using the definition of an even number again, n^2 is even.

$\diamondsuit$

Typical Proof. Suppose n is even. By the definition of an even integer, there is an integer m such that $n = 2m$. So

$$n^2 = (2m)^2 = 4m^2 = 2(2m^2).$$

By the definition of an even integer, n^2 is even.

$\diamondsuit$

Example Theorem 3.6. If n and m are perfect squares then nm is a perfect square.

Detailed Proof.

1. Suppose that n and m are perfect squares.

Known variables:	n, m
Know:	n is a perfect square.
Know:	m is a perfect square.
Goal:	nm is a perfect square.

2. By the definition of a perfect square, there exist integers x and y such that $n = x^2$ and $m = y^2$.

Known variables:	n, m, x, y
Know:	$n = x^2$.
Know:	$m = y^2$.
Goal:	nm is a perfect square.

3. Replacing n by x^2 and m by y^2, $nm = x^2y^2 = (xy)^2$.

Known variables:	n, m, x, y
Know:	$n = x^2$.
Know:	$m = y^2$.
Know:	$nm = (xy)^2$.
Goal:	nm is a perfect square.

4. So $nm = z^2$ where $z = xy$. Using the definition of a perfect square again, nm is perfect square.

♢

Typical Proof. Suppose that n and m are perfect squares. By the definition of a perfect square, there exist integers x and y such that $n = x^2$ and $m = y^2$. Replacing n by x^2 and m by y^2,

$$nm = x^2y^2 = (xy)^2.$$

So nm is a perfect square.

♢

3.5.1 Using the Contrapositive

You can prove any theorem by proving an equivalent mathematical statement. For example, you can prove $A \to B$ by proving equivalent formula $\neg B \to \neg A$, which is called the *contrapositive* of $A \to B$. Here is an example.

Example Theorem 3.7. Suppose n is an integer. If $3n + 2$ is odd, then n is odd.

Detailed Proof. We prove the contrapositive: If n is not odd then $3n + 2$ is not odd.

1. We know that an integer x is even if and only if x is not odd. So what we want to prove is equivalent to: If n is even then $3n + 2$ is even.

Known variables:	n
Goal:	If n is even then $3n + 2$ is even.

2. Suppose that n is even.

Known variables:	n
Know:	n is even.
Goal:	$3n + 2$ is even.

3. By the definition of an even integer, there exists an integer m such that $n = 2m$.

Known variables:	n, m
Know:	$n = 2m$.
Goal:	$3n + 2$ is even.

4. $3n + 2 = 3(2m) + 2 = 6m + 2 = 2(3m + 1)$.

Known variables:	n, m
Know:	$n = 2m$.
Know:	$3n + 2 = 2(3m + 1)$.
Goal:	$3n + 2$ is even.

5. Using the definition of an even integer again, $3n + 2$ is even because $3n + 2 = 2z$ where $z = 3m + 1$.

♢

Typical Proof. We prove the contrapositive: If n even then $3n + 2$ is even.

Suppose n is even. Then there exists an integer m such that $n = 2m$.

$$3n + 2 = 3(2m) + 2 = 6m + 2 = 2(3m + 1).$$

Since $3n + 2$ is twice an integer, $3n + 2$ is even.

♢

3.6 Proving and Using (A and B)

To prove $A \wedge B$, prove A and prove B.

If you know that $A \wedge B$ is true, then you know that A is true and you know that B is true.

3.7 Proving and Using not(A)

To prove $\neg(A)$, you typically use DeMorgan's laws and the laws for negating quantified formulas to push the negation inward. For example, to prove $\neg(A \wedge B)$, you prove equivalent formula $\neg A \vee \neg B$. To prove $\neg(\forall x A)$, you prove equivalent formula $\exists x(\neg A)$.

The same principle applies when you already know $\neg(A)$. For example, if you know $\neg(A \to B)$, you can conclude equivalent formula $A \wedge \neg B$. You write that down as an additional known fact.

3.8 Proving and Using (A or B)

To prove $A \vee B$, you usually prove one of the equivalent formulas $\neg A \to B$ or $\neg B \to A$.

Suppose that you know that $A \vee B$ is true and you want to use that to show that C is true. That is, you want to show that $A \vee B \to C$ is true. You typically prove equivalent formula

$$A \to C \ \wedge \ B \to C.$$

That is called *proof by cases*. First, you assume that A is true and show that C is true. Next, you assume that B is true and show that C is true. See Section 3.13.

Proof by cases requires you to do each case in a silo. Assumptions made while proving $A \to C$ cannot be used when proving $B \to C$. When you start to prove $B \to C$, your knowledge reverts to what it was when you were just about to start to prove $A \to C$.

3.9 Proving and Using Existential Statements

To prove that something exists, produce it. That is called a *constructive existence proof.*

Example Theorem 3.8. There exists an integer n where n is even and n is prime.

Proof. Choose $n = 2$. Notice that n is even and n is prime.

$\diamondsuit$

3.9.1 Using Existential Knowledge

Sometimes, instead of needing to prove $\exists x P(x)$, you already know $\exists x P(x)$. What do you do? You ask somebody else to give you a value x so that $P(x)$ is true. It is not necessary for you to say how to find x. We will encounter many examples of that.

3.10 Proving Universal Statements

To prove $\forall x P(x)$, prove $P(x)$ for an *arbitrary* value of x.

That does not mean that you can choose the value of x. Rather, someone else chooses x and you must prove that $P(x)$ is true for that value of x. Think of it as a challenge. You say to someone else, give me any value of x that you like. I will prove that $P(x)$ is true. In mathematics, *arbitrary* always means a value chosen by someone else.

We have actually used this idea above. When the statement of a theorem involves unbound (unquantified) variables, it is assumed to be saying that the statement is true for all values of those variables. Here is the first proof above with the quantifier explicit. The universe of discourse is the set of all integers.

Example Theorem 3.9. $\forall n(n$ is even $\rightarrow n^2$ is even).

Detailed Proof.

1. Ask someone else to select an arbitrary integer n. (We cannot assume anything about n except that it belongs to the universe of discourse.) We must prove: (n is even $\rightarrow n^2$ is even) for that n.

Known variables:	n
Goal:	n is even $\rightarrow n^2$ is even.

2. Suppose that n is even.

Known variables:	n
Know:	n is even.
Goal:	n^2 is even.

3. By the definition of an even integer, there exists an integer m such that $n = 2m$.

Known variables:	n
Know:	$\exists m(n = 2m)$.
Goal:	n^2 is even.

4. Ask someone else to provide the integer m that is asserted to exist.

Known variables:	n, m
Know:	$n = 2m$.
Goal:	n^2 is even.

5. Since $n = 2m$, $n^2 = (2m)^2 = 4m^2 = 2(2m^2)$.

Known variables:	n, m
Know:	$n = 2m$.
Know:	$n^2 = 2(2m^2)$.
Goal:	n^2 is even.

6. So $n = 2(x)$ where $x = 2m^2$. Using the definition of an even number again, n is even.

$\diamondsuit$

Typical Proof. Let n be an arbitrary even integer. By the definition of an even integer, there exists an integer m such that $n = 2m$. So

$$n^2 = (2m)^2 = 4m^2 = 2(2m^2).$$

Evidently, n^2 is even.

$\diamondsuit$

3.10.1 Proof by Contradiction

You can prove any theorem by proving an equivalent theorem. We have seen propositional tautology

$$p \leftrightarrow (\neg p \rightarrow \mathbf{F}).$$

That is, to prove p, assume that p is false and prove that **F** is true. Typically, you use the tautology

$$(q \wedge \neg q) \leftrightarrow \mathbf{F}$$

to prove **F** by proving a statement and its negation. That is called *proof by contradiction*. Let's use proof by contradition to reprove a theorem that we proved above.

Example Theorem 3.10. For every integer n, if $3n + 2$ is odd, then n is odd.

Detailed Proof.

1. Reasoning by contradiction, we can assume the theorem is false and prove **F**. That is:

Know:	$\neg\forall n(3n + 2 \text{ is odd } \rightarrow n \text{ is odd})$.
Goal:	**F**.

2. We can push the negation across the quantifier using valid formula $\neg\forall xA \leftrightarrow \exists x(\neg A))$.

Know:	$\exists n(\neg(3n + 2 \text{ is odd } \rightarrow n \text{ is odd}))$.
Goal:	**F**.

3. Now use the tautology that $\neg(p \rightarrow q) \leftrightarrow p \wedge \neg q)$.

Know:	$\exists n(3n + 2 \text{ is odd } \wedge n \text{ is even})$.
Goal:	**F**.

4. Ask somebody else to select an integer n such that $3n + 2$ is odd and n is even.

Known variables:	n
Know:	$3n + 2$ is odd.
Know:	n is even.
Goal:	**F**.

5. By the definition of an even integer, saying that n is even is equivalent to saying that there exists an integer m such that $n = 2m$. (Existential information is useful because it allows you to get something in hand, as is done in the next step. So you often want to exploit existential information.)

Known variables:	n
Know:	$3n + 2$ is odd.
Know:	$\exists m(n = 2m)$.
Goal:	**F**.

6. Since we know that an integer m exists such that $n = 2m$, we can ask somebody else to give us such an m. Let's do that.

Known variables:	n, m
Know:	$3n + 2$ is odd.
Know:	$n = 2m$.
Goal:	**F**.

7. Since we know that $n = 2m$, it seems reasonable to substitute $2m$ for n in expression $3n + 2$ to see what we get. Doing that gives

$$3n + 2 = 3(2m) + 2 = 6m + 2 = 2(3m + 1).$$

So $3n + 2$ is even. Recording that:

Known variables:	n, m
Know:	$3n + 2$ is odd.
Know:	$n = 2m$.
Know:	$3n + 2$ is even.
Goal:	**F**.

8. But $3n + 2$ cannot be both even and odd. Formula ($3n + 2$ is odd $\wedge$ $3n + 2$ is even) is equivalent to **F**. So we have concluded that **F** is true and we are done.

$\diamondsuit$

Typical Proof. By contradiction. Assume there exists an n such that $3n+2$ is odd but n even. Since n is even, there exists an integer m so that $n = 2m$. So

$$3n + 2 = 3(2m) + 2 = 6m + 2 = 2(3m + 1).$$

That means $3n + 2$ is even, contradicting the assumption that $3n + 2$ is odd.

$\diamondsuit$

3.11 Proving $\forall x(\exists y(A))$

It is common to encounter theorems whose general form is $\forall x(\exists y P(x, y))$. The proof usually involves finding an algorithm. For any x, the algorithm must find a y so that $P(x, y)$ is true. Here is an example.

Example Theorem 3.11. For all real numbers x and y, if x and y are both rational numbers then $x + y$ is also a rational number.

Detailed Proof.

1. Ask someone else to select arbitrary real numbers of x and y.

Known variables:	x, y
Goal:	If x and y are rational then $x + y$ is rational.

2. Assume that x and y are rational.

Known variables:	x, y
Know:	x is rational.
Know:	y is rational.
Goal:	$x + y$ is rational.

3. Our knowledge involves the term *rational*. We need to know what that means. From the definition of a rational number, there must exist integers a and b where $b \neq 0$ and $x = a/b$; and there must exist integers c and d where $d \neq 0$ and $y = c/d$. (Notice that different names are chosen for different things.)

Known variables:	x, y, a, b, c, d
Know:	a, b, c and d are integers.
Know:	$b \neq 0$.
Know:	$d \neq 0$.
Know:	$x = a/b$.
Know:	$y = c/d$.
Goal:	$x + y$ is rational.

4. Since the goal is to show that $x + y$ is rational, let's replace x by a/b and replace y by c/d in expression $x + y$. Since b and d are nonzero,

$$x + y = a/b + c/d = ad/bd + bc/bd = (ad + bc)/bd.$$

Known variables:	x, y, a, b, c, d
Know:	a, b, c and d are integers.
Know:	$b \neq 0$.
Know:	$d \neq 0$.
Know:	$x = a/b$.
Know:	$y = c/d$.
Know:	$x + y = (ad + bc)/bd$.
Goal:	$x + y$ is rational.

5. But we have shown that $x + y$ is the ratio of integers $ad + bc$ and bd. Since neither b nor d is 0, bd cannot be 0. So $x + y$ is rational, by the definition of a rational number.

   The algorithm that has been employed here is for adding two fractions.

$\diamondsuit$

Typical Proof. Let x and y be arbitrary rational numbers. By the definition of a rational number, there exists integers a, b, c and d ($b \neq 0$ and $d \neq 0$) such that $x = a/b$ and $y = c/d$. Then

$$x + y = a/b + c/d = ad/bd + bc/bd = (ad + bc)/bd.$$

Since $x + y$ is the ratio of two integers, $x + y$ is rational. (You can observe that $bd \neq 0$ since the product of two nonzero numbers is nonzero.)

3.12 Proving (A If and Only If B)

There are two commonly used ways of proving $A \leftrightarrow B$.

3.12.1 Using Direct Equivalences

You can treat $\leftrightarrow$ in a way similar to the way you treat $=$ in algebraic equations, performing equivalence-preserving manipulations. Let's use that approach to prove the law of the contrapositive.

Example Theorem 3.12. $p \to q \leftrightarrow \neg q \to \neg p$.

Proof.

$$\begin{array}{lcll} \neg q \to \neg p & \leftrightarrow & \neg(\neg q) \vee \neg p & \text{(defn of } \to) \\ & \leftrightarrow & q \vee \neg p & \text{(double negation)} \\ & \leftrightarrow & \neg p \vee q & \text{(commutative law of } \vee) \\ & \leftrightarrow & p \to q & \text{(defn of } \to) \end{array}$$

3.12.2 Proving Two Implications

Sometimes it is preferable to use the fact that $A \leftrightarrow B$, is equivalent to $(A \to B) \wedge (B \to A)$ and to prove $A \to B$ and $B \to A$ separately.

Example Theorem 3.13. For every integer n, n is odd if and only if n^2 is odd.

Detailed Proof.

1. It suffices to prove

$$\forall n((n \text{ is odd } \to n^2 \text{ is odd}) \;\wedge\; (n^2 \text{ is odd } \to n \text{ is odd})).$$

That gives two goals. We use tautology $\forall x(A \wedge B) \leftrightarrow (\forall x A \wedge \forall x B)$ and change the variable names so that we can look at the two parts separately without variables from one interfering with the other.

Goal (1):	$\forall n(n \text{ is odd } \to n^2 \text{ is odd})$.
Goal (2):	$\forall m(m^2 \text{ is odd } \to m \text{ is odd})$.

2. Ask someone else to choose arbitrary values of m and n.

Known variables:	n, m
Goal (1):	n is odd $\rightarrow n^2$ is odd.
Goal (2):	m^2 is odd $\rightarrow m$ is odd.

3. Goal (2) is equivalent to its contrapositive, m is even $\rightarrow m^2$ is even. We proved that as Theorem 3.1. That only leaves Goal (1). (We still know goal (2), of course, but we can always discard known information to simplify.)

Known variables:	n
Goal (1):	n is odd $\rightarrow n^2$ is odd.

4. To prove Goal (1), assume that n is odd.

Known variables:	n
Know:	n is odd.
Goal (1):	n^2 is odd.

5. Since n is odd, there exists an integer k so that $n = 2k + 1$.

Known variables:	n
Know:	$\exists k(n = 2k + 1)$.
Goal (1):	n^2 is odd.

6. Ask someone else to provide a value k such that $n = 2k + 1$.

Known variables:	n, k
Know:	$n = 2k + 1$.
Goal (1):	n^2 is odd.

7. Since $n = 2k + 1$,

$$n^2 = (2k + 1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1.$$

Since $n^2 = 2z + 1$ for $z = 2k^2 + 2k$, it is evident that n^2 is odd.

◊

Typical Proof.

(a) (n is odd $\to n^2$ is odd) Assume that n is odd. By the definition of an odd integer, there is an integer k such that $n = 2k + 1$. So

$$n^2 = (2k + 1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1.$$

By the definition of an odd integer, n^2 is odd.

(b) (n^2 is odd $\to n$ is odd) This is equivalent to (n is even $\to n^2$ is even), which we profed earlier as Theorem 3.1.

◊

3.13 Proof by Cases

Proof by cases involves proving two or more statements. You must be careful that assumptions made during one of those cases are not still in place when proving another one. Think of this is similar to calling a function in a program. Each time a function is called, a new frame is created, so that calling $f(3)$ does not interfere with a later call to $f(4)$.

Example Theorem 3.14. For every integer n, $n^2 \geq n$.

Detailed Proof.

1. Ask someone to select an arbitrary integer n.

Known variables:	n
Know:	n is an integer
Goal:	$n^2 \geq n$.

2. Let's break proving the goal into three cases: $n = 0$, $n > 0$ and $n < 0$.

Known variables:	n
Know:	n is an integer
Goal (1):	$n = 0 \to n^2 \geq n.$
Goal (2):	$n > 0 \to n^2 \geq n.$
Goal (3):	$n < 0 \to n^2 \geq n.$
Goal (4):	$n^2 \geq n.$

3. Goal (1) is clearly true since $0^2 \geq 0$. Let's record it among the known facts.

Known variables:	n
Know:	n is an integer
Know (1):	$n = 0 \to n^2 \geq n.$
Goal (2):	$n > 0 \to n^2 \geq n.$
Goal (3):	$n < 0 \to n^2 \geq n.$
Goal (4):	$n^2 \geq n.$

4. Goal (2) is an implication, so we should assume that $n > 0$ and prove that $n^2 \geq n$. But let's prove that as a separate subproof. Knowledge and goals that are local to the proof of goal (2) are numbered 2.1, 2.2, etc., and they can only be used to establish goal (2).

Known variables:	n
Know:	n is an integer
Know (1):	$n = 0 \to n^2 \geq n.$
Goal (2):	$n > 0 \to n^2 \geq n.$
Goal (3):	$n < 0 \to n^2 \geq n.$
Goal (4):	$n^2 \geq n.$
Know (2.1):	$n > 0$
Goal (2.1):	$n^2 \geq n$

5. Since $n > 0$ is an integer, it must be the case that $n \geq 1$.

Known variables:	n
Know:	n is an integer
Know (1):	$n = 0 \to n^2 \geq n.$
Goal (2):	$n > 0 \to n^2 \geq n.$
Goal (3):	$n < 0 \to n^2 \geq n.$
Goal (4):	$n^2 \geq n.$
Know (2.1):	$n \geq 1$
Goal (2.1):	$n^2 \geq n$

Multiplying both sides of fact (2.1) by n preserves the inequality because $n > 0$. That gives $n \cdot n \geq n \cdot 1$, or equivalently, $n^2 \geq n$.

Known variables:	n
Know:	n is an integer
Know (1):	$n = 0 \to n^2 \geq n.$
Goal (2):	$n > 0 \to n^2 \geq n.$
Goal (3):	$n < 0 \to n^2 \geq n.$
Goal (4):	$n^2 \geq n.$
Know (2.1):	$n \geq 1$
Know (2.2):	$n^2 \geq n$
Goal (2.1):	$n^2 \geq n$

6. We have succeeded in proving goal (2). Notice that fact (2.2) cannot be used to establish goal (4) since it depends on the assumption that $n > 0$.

   We can move goal (2) into our knowledge. But we must also throw out parts that were local to the proof of goal (2).

Known variables:	n
Know:	n is an integer
Know (1):	$n = 0 \to n^2 \geq n.$
Know (2):	$n > 0 \to n^2 \geq n.$
Goal (3):	$n < 0 \to n^2 \geq n.$
Goal (4):	$n^2 \geq n.$

7. Now we need to prove goal (3). Assume that $n < 0$. But the square of any number is nonnegative. It follows that $n^2 \geq 0 > n$ when $n < 0$, and we can move goal (3) into what we know.

Known variables:	n
Know:	n is an integer
Know (1):	$n = 0 \rightarrow n^2 \geq n$.
Know (2):	$n > 0 \rightarrow n^2 \geq n$.
Know (3):	$n < 0 \rightarrow n^2 \geq n$.
Goal (4):	$n^2 \geq n$.

8. Propositional formula

$$(p \rightarrow s) \wedge (q \rightarrow s) \wedge (r \rightarrow s) \wedge (p \vee q \vee r)) \rightarrow s$$

is a tautology. That means known facts (1), (2) and (3) imply goal (4).

♢

Typical Proof. The proof is by cases ($n = 0$, $n > 0$ and $n < 0$).

Case 1 ($n = 0$). Then $n^2 \geq n$ because $0^2 \geq 0$.

Case 2 ($n > 0$). The smallest positive integer is 1, so $n > 0$ implies $n \geq 1$. Multiplying both sides of inequality $n \geq 1$ by positive number n gives $n^2 \geq n$.

Case 3 ($n < 0$). $n^2 \geq 0$ for all numbers n. Since, in this case, n is negative, clearly $n^2 \geq n$.

♢

prev **next**