Proof

$$\sum_{k=0}^{n} x^k = \frac{x^{n+1}-1}{x-1} \quad \text{when } x \neq 1$$

Let

$$(1) \quad S = \sum_{k=0}^{n} x^k = 1 + x + x^2 + \dots + x^{n-1} + x^n$$

then

$$(2) \quad xS = x + x^2 + x^3 + \dots + x + x^{n+1}$$

Subtracting (1) from (2) yields

$$xS - S = x^{n+1} - 1$$ (All terms cancel except the 1 in (1) and x^{n+1} in (2))

$$S(x-1) = x^{n+1} - 1$$

$$S = \frac{x^{n+1}-1}{x-1}$$

and when $0 < x < 1$, $\lim_{n \to \infty} S = \frac{-1}{x-1} = \frac{1}{1-x}$

(because x^{n+1} goes to 0)

Proof that Regularity Condition always holds for Master Theorem Case 3 when $f(n) = n^K$

Case 3 means

$$f(n) = \Omega(n^{\log_b a + \epsilon}) \quad \epsilon > 0$$

If $f(n) = n^K$ then we have

$$n^K > n^{\log_b a} \quad \text{Since } K > \log_b a \text{ and } n \geq 1$$

(constant c from Ω dfn. goes away)

using $K > \log_b a$ if we apply exponentiation by b to both sides we get

$$b^K > a \quad \text{or} \quad \frac{a}{b^K} < 1$$

multiplying both sides by n^K yields

$$a\left(\frac{n}{b}\right)^K < n^K$$

$$\text{or } a f\left(\frac{n}{b}\right) < f(n)$$

Take $c = 1$ and

thus $a f\left(\frac{n}{b}\right) \leq c f(n)$ (Regularity condition holds)